k -TO-1 FUNCTIONS BETWEEN GRAPHS WITH FINITELY MANY DISCONTINUITIES

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ABSTRACT. This paper classifies which pairs of graphs G and H and which integers $k \geq 2$ allow k -to-1 functions from G onto H with only finitely many discontinuities.

1. Introduction. It has long been known that there are no 2-to-1 continuous functions defined on graphs with odd Euler number [3], nor are there k -to-1 continuous functions onto any dendrite [4], and hence any acyclic graph, for $k > 1$. There are many other results on continuous k -to-1 maps; see [1, 2, 5, 6, 12–15].

More recently, studies have been made to determine which spaces allow k -to-1 functions (to them or from them) with only a finite number of discontinuities. For instance, odd-Euler graphs do not support 2-to-1 functions with only finitely many discontinuities [7] and for $k > 2$, Kellum and Katsuura proved that every k -to-1 function from $[0, 1]$ onto $[0, 1]$ has infinitely many discontinuities [10]. This latter result was generalized in [9] to functions from continua onto dendrites. See also [8, 11].

This paper answers the question: given an integer $k > 1$, for which finite graphs (1-complexes) G and H does there exist a k -to-1 function from G onto H with only finitely many discontinuities? The principal feature of a dendrite that prohibits its being the image of a k -to-1 function is its lack of simple closed curves, and for graphs, the number of simple closed curves in G and in H completely determines the answer to the above question.

There are a lot more questions about k -to-1 functions, discontinuous or continuous, than there are answers, and here are a few:

QUESTION 1 (R. LEVY [12]). Is a circle the only 2-to-1 continuous image of $[0, 1]$?

QUESTION 2. Can the results of this paper be generalized to 2-dimensional finite complexes?

QUESTION 3. For which k and for which continua X and Y are uncountably many discontinuities required of any k -to-1 function from X onto Y ?

2. Definitions and Lemma 0. A function with only finitely many discontinuities will be called a *FD-function*. The continuous case is not excepted.

If G is a finite graph whose n_1 endpoints and junction points separate G into m_1 disjoint open arcs, and if another set in G with n points separates G into m

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disjoint open arcs, then the n points contain the n_1 points plus a few from the m_1 arcs, and $m - n = m_1 - n_1$. Thus the following is well defined:

A graph G is j -multicyclic if there are n points in G whose complement in G is the union of m disjoint open arcs and $j = m - n + 1 > 0$. If $j = 1$, G is called *unicyclic*.

Note that if G is embedded in the plane, then G is j -multicyclic iff G has j bounded complementary domains.

The following notion drastically reduces the number of cases. For instance, it permits every acyclic graph to be treated as the interval $[0, 1]$.

Suppose the space X is the union of n disjoint subsets, $X = X_1 + X_2 + \cdots + X_n$, and Y is also the union of n disjoint subsets $Y = Y_1 + Y_2 + \cdots + Y_n$. Suppose further that there is a function h from X onto Y whose restriction to X_i is a homeomorphism onto Y_i , for each $i = 1, 2, \dots, n$. Then X is a *rearrangement* of Y if for each function f defined on Y , f has finitely many discontinuities iff fh (defined on X) has finitely many discontinuities.

For instance, the θ curve is a rearrangement of $(0, 1) + (2, 3) + (4, 5) + \{6, 8\}$, and so is $(0, 1)$, by thinking of $(0, 1)$ as $(0, \frac{1}{3}) + (\frac{1}{3}, \frac{2}{3}) + (\frac{2}{3}, 1) + \{\frac{1}{3}, \frac{2}{3}\}$. Note that rearrangement is an equivalence relation.

The next lemma will be used often without specific mention:

LEMMA 0. *If G is an acyclic graph, then $[0, 1]$ is a rearrangement of G ; if G is unicyclic, then $[0, 1)$ is a rearrangement of G ; and if G is j -multicyclic with $j > 1$, then the disjoint union of $j - 1$ open intervals is a rearrangement of G .*

PROOF. For all three cases, let n denote the number of endpoints and junction points of G plus one (to take care of $G = S^1$), let m denote the number of open arcs in the complement of these n points, and let $j = m - n + 1$. Then $T = (1, 2) + (2, 3) + \cdots + (m, m + 1) + \{1, 2, \dots, n\}$ is a rearrangement of G .

Since every acyclic graph has $n = m + 1$, $[1, m + 1]$ is a rearrangement of T and hence of G . If G is unicyclic then $j = 1$, so $n = m$ and $[1, m + 1]$ is a rearrangement of T and of G . Now suppose $j > 1$. If $n < m$, then m open intervals plus n points rearrange as $m - n$ open intervals. Since j is greater than 1, $n = m + 1 - j$ is less than m , so that G rearranges as the disjoint union of $m - (m + 1 - j)$ or $j - 1$ open intervals.

3. Reference facts from other papers:

1. [7] There is no 2-to-1 function on $[0, 1]$ or on $(0, 1)$ with finitely many discontinuities, with Hausdorff image.

2. [9] There is no k -to-1 function from any continuum onto $[0, 1]$ with only finitely many discontinuities, for $k > 1$.

3. [11] There is no k -to-1 function from $(0, 1)$ onto $[0, 1]$ with only finitely many discontinuities, for $k > 1$.

4. [11] For every $k > 1$ there is a k -to-1 function from $(0, 1)$ onto $(0, 1)$ with only finitely many discontinuities.

5. [10, 11] For each $k > 2$, there is a k -to-1 function from $[0, 1]$ onto $(0, 1)$ with only finitely many discontinuities.

6. [6, 10] For each $k > 2$, there is a k -to-1 function from $[0, 1]$ onto $[0, 1]$ with only finitely many discontinuities.

4. Lemmas. These lemmas will be used in the next section on the rearranged graphs.

LEMMA 1. *There is no 2-to-1 function from $[0, 1)$ onto the union of t disjoint open intervals with only finitely many discontinuities.*

PROOF. Suppose there is such a function f . Let $D \subseteq [0, 1)$ be the set of discontinuities of f plus the number 0. Then the t open intervals of the image minus the n points of $f(D)$ is the union of $t + n$ disjoint open intervals, $J_1 + \cdots + J_{t+n}$. Since f is 2-to-1, $f^{-1}(f(D))$ has $2n$ points, $0 = x_1 < \cdots < x_{2n}$. Let $x_{2n+1} = 1$. Since f restricted to each (x_i, x_{i+1}) for $i = 1, 2, \dots, 2n$ is continuous, it maps each into only one J_j . For each $j = 1, 2, \dots, t + n$, let $h(j)$ denote the number of open intervals (x_i, x_{i+1}) that map into J_j . Then

1. $h(1) + h(2) + \cdots + h(n + t) = 2n$ and
2. $h(1) + h(2) + \cdots + h(n + t) \geq 2n + 2t$.

The first is true because there are $2n$ of the (x_i, x_{i+1}) intervals, the second is true because $h(j) \geq 2$ for each j by reference fact 1. This contradiction completes the proof. (The proof of this lemma is a variation of a proof in [10].)

LEMMA 2. *There is no k -to-1 function, $k > 1$, from the disjoint union of m open intervals onto $[0, 1)$ with only finitely many discontinuities.*

PROOF. Suppose there is such a function f . Let q be a positive integer such that $qm \geq k$. Then q copies of f take mq open intervals onto $[0, 1) + [1, 2) + \cdots + [q-1, q) = [0, q)$. This function can be extended to a k -to-1 FD -function taking the qm open intervals plus k new points onto $[0, q]$ by sending the new points to q . But qm open intervals plus k points can be rearranged as a continuum since $k \leq qm + 1$. This contradicts reference fact 2.

LEMMA 3. *If $k > 1$ and $0 < m \leq k$, then there is a k -to-1 function from the union of m disjoint open intervals onto $(0, 1)$ with only finitely many discontinuities, except for the one case $m = 1$ and $k = 2$.*

PROOF. If $m \leq k$, then $r = k - m \geq 0$. If $r \neq 1$, then there is a $(r + 1)$ -to-1 FD -function from the first open interval in the domain onto $(0, 1)$ by reference fact 4. Then each of the remaining $m - 1$ open intervals of the domain can be mapped 1-to-1 onto $(0, 1)$. The union of these maps is the required k -to-1 FD -function. If $r = 1$ then $m = k - 1$ so the m open intervals plus k new points rearranges to $[0, 1]$, and the image $(0, 1)$ plus 1 new point is $[0, 1)$. By reference fact 6, if $k \neq 2$, there is a k -to-1 FD -function f from $[0, 1]$ onto $[0, 1)$. (If $k = 2$ then $m = k - 1 = 1$ is the one excluded case.) Then $[0, 1]$ minus the k points of $f^{-1}(0)$ maps via f to $(0, 1)$. But $[0, 1]$ minus k points rearranges to $k - 1$ intervals so the restricted f is the required function.

LEMMA 4. *If $m > k > 1$, then there is no k -to-1 function from the union of m disjoint intervals onto $(0, 1)$ with only finitely many discontinuities.*

PROOF. Since $m > k$, $m - k$ is positive. Let q be an integer such that $q > (k - 1)/(m - k)$. Then $qm + 1 > k(q + 1)$. Suppose there is a k -to-1 FD -function from the m open intervals onto $(0, 1)$. Then the union, h , of q copies of this function maps the union of qm open intervals onto the union of q open intervals. But since

$qm + 1 > k(q + 1)$, the qm open intervals plus $k(q + 1)$ new points can be rearranged into a continuum M . Then h extends to M by mapping the $k(q + 1)$ new points to $q + 1$ new points in the range. But the new image of q open intervals plus $q + 1$ points rearranges as $[0, 1]$ contradicting reference fact 2.

LEMMA 5. *If $k > 2$ then there is a k -to-1 function from $[0, 1)$ onto $(0, 1)$ with only finitely many discontinuities.*

PROOF. Let $k > 2$. Let g and h be any 3-to-1 FD -functions from $[0, 1]$ onto $[0, 1)$ and from $(0, 1)$ onto $(0, 1)$ respectively. See reference facts 6 and 4. The functions for $k = 3, 4$ and 5 will be painfully constructed and then all other functions are unions of these three maps.

(a) Define the 3-to-1 FD -function f_3 from $[0, 2)$ onto $(0, 2)$ by

$$\begin{aligned} f_3(x) &= 1 - g(x) & \text{if } x \in [0, 1] \\ &= h(x) + 1 & \text{if } x \in (1, 2). \end{aligned}$$

(b) Define the 4-to-1 FD -function f_4 from $[0, 3)$ onto $(0, 2)$ by

$$\begin{aligned} f_4(x) &= h(x) & \text{if } x \in (0, 1), \\ &= h(x) + 1 & \text{if } x \in (1, 2), \\ &= 2(x - 2) & \text{if } x \in (2, 3), \\ &= 1 & \text{if } x = 0, 1 \text{ or } 2. \end{aligned}$$

(c) Define the 5-to-1 FD -function f_5 from $[0, 5)$ onto $(0, 3)$ by:

$$\begin{aligned} f_5(x) &= h(x) & \text{if } x \in (0, 1), \\ &= h(x) + 1 & \text{if } x \in (1, 2), \\ &= g(x) + 2 & \text{if } x \in [2, 3], \\ &= 3(x - 3) & \text{if } x \in (3, 4), \\ &= 3(x - 4) & \text{if } x \in (4, 5), \\ &= 1 & \text{if } x = 0, 1 \text{ or } 4. \end{aligned}$$

Now, to avoid getting bogged down in endpoint values, think of each f_k , $k = 3, 4, 5$, as a generic k -to-1 FD -function from a half open interval onto an open interval.

If $k > 5$, write $k = 3q + 4r + 5t$ for nonnegative integers q, r and t . Inductively, it is clear that $k = 3, 4, 5$ can be so written and that k can be if $k - 3$ can be.

The domain $[0, 1)$ can be rearranged as $q + r + t$ disjoint half-open intervals. Map the first q of them onto $(0, 1)$ via f_3 , the next r of them onto $(0, 1)$ via f_4 , and the last t of them onto $(0, 1)$ via f_5 . The union of these $r + s + t$ functions is a k -to-1 FD -function from $[0, 1)$ onto $(0, 1)$.

5. Theorems and proofs.

THEOREM 1. *If G is a $(m + 1)$ -multicyclic graph and H is a $(t + 1)$ -multicyclic graph, then there is a k -to-1 function from G onto H with only finitely many discontinuities:*

- (A) *iff $m \leq kt$, if $k > 2$, and*
- (B) *iff $m = 2t$, if $k = 2$.*

PROOF. 1. *A reduction.*

Suppose $m \geq k + 1$ and $t > 1$. Then neither G nor H is unicyclic. Hence G can be rearranged as the disjoint union of m open intervals and H as t open intervals. There is a k -to-1 FD -function from m open intervals onto t open intervals iff there is one from m open intervals plus k new points onto t open intervals plus 1 new point. But m open intervals and k points rearrange to $m - k$ open intervals, and t open intervals and one point rearrange to $t - 1$ open intervals. If q is the largest integer such that $m' = m - qk > 0$ and $t' = t - q > 0$ then $m' \leq k$ or $t' \leq 1$ and there is a FD -function from G onto H iff there is one from m' open intervals onto t' open intervals. Also note that $m \leq kt$ iff $m' \leq kt'$ and $m = 2t$ iff $m' = 2t'$. Hence it can be assumed that

$$(1) \quad t \leq 1 \quad \text{or} \quad m \leq k.$$

2. *Proof of (A) and (B) if G and H are both unicyclic.* G can be rearranged as $[0, k)$ and H can be rearranged as $[0, 1)$. Define the k -to-1 FD -function f from $[0, k)$ onto $[0, 1)$ by $f(x) = x - \lfloor x \rfloor$.

3. *Proof of (A).*

($\leftarrow$) Let $m \leq kt$ and $k > 2$.

Then $t = 0$ implies $m = 0$, done in part 2, so by the reduction (1), $t = 1$ or $m \leq k$. But $t = 1$ and $m \leq kt$ means $m \leq k$, either way.

A further reduction. Suppose $t > 1$. Write G as $[0, t)$ or as $(0, t) + I_2 + \cdots + I_m$ depending on whether G is unicyclic or not. Map each $[n, n + 1)$ k -to-1 onto the n th open interval of H using Lemma 5, for $n = 1, 2, \dots, t - 1$. This leaves only one open interval in H not mapped onto and G is essentially the same, $[0, 1)$ or $(0, 1) + I_2 + \cdots + I_m$. In the former case use Lemma 5 to map $[0, 1)$ onto the rest of H , and in the latter case map the m intervals of G onto $(0, 1)$ using Lemma 3.

($\rightarrow$) Let $m > kt$, $k > 2$.

Since $m > kt$, G is not unicyclic. Also, $m > kt$ plus the reduction (1) $m \leq kt$ or $t \leq 1$ implies $t \leq 1$. If $t = 0$, Lemma 2 is used, and if $t = 1$, Lemma 4 is used.

4. *Proof of (B).*

($\leftarrow$) Suppose $m = 2t$ and $k = 2$.

Since $m = 2t$ and the $m = t = 0$ case was done in part 2, we assume that G can be rearranged as $2t$ open intervals and H as t open intervals. But then G obviously maps 2-to-1 onto H .

($\rightarrow$) Suppose $m \neq 2t$ and $k = 2$.

If $t = 0$, Lemma 2 is used, since $m \neq 0$.

If $m = 0$, Lemma 1 is used, since $t \neq 0$.

The case $m = 1$ is covered by reference fact 1.

Hence, by the reduction (1) we can assume that $t = 1$ or $m = 2$.

If $t = 1$, then $m \neq 2$ since $m \neq 2t$, and $m > 2$ is covered by Lemma 4.

If $t > 1$ and $m = 2$, then two new points can be added to G and rearranged as $[0, 1)$, and H with one new point rearranges as $t - 1$ open intervals. This case is covered by Lemma 1.

This completes the proof of Theorem 1.

THEOREM 2. *Suppose G and H are finite graphs, one of them is acyclic, and $k > 1$. Then*

(1) *If H is acyclic, then there is no k -to-1 function from G onto H with only finitely many discontinuities.*

(2) *If G is acyclic and H is any multicyclic graph, then there is a k -to-1 function from G onto H with only finitely many discontinuities iff $k > 2$.*

PROOF. 1. If H is an acyclic graph then H rearranges into $[0, 1]$ and there is no k -to-1 FD -function from any graph G onto H by reference fact 2.

2. If $k = 2$, and G is acyclic, then there is no 2-to-1 FD -function defined on G at all, by reference fact 1.

3. If $k > 2$, G is acyclic, and H is unicyclic, then there is a k -to-1 FD -function from $[0, 1]$ onto $[0, 1)$ by reference fact 6.

4. If $k > 2$, G is acyclic and H is $(t + 1)$ -multicyclic with $t > 0$, then H can be rearranged as $J_1 + \cdots + J_t$, open intervals, and G as $[0, t]$. There is a k -to-1 function (from reference fact 5) from $[0, 1]$ onto J_1 and for each $i = 2, 3, \dots, t$, there is a k -to-1 function (from Lemma 5) from $(i - 1, i]$ onto J_i . The union of these functions is a k -to-1 FD -function from G onto H .

ADDENDUM. The answer to Question 1 can be found in [14].

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