THE CONVOLUTION OF RADON MEASURES

H. KHARAGHANI

(Communicated by Richard R. Goldberg)

ABSTRACT. The convolution of a pair of bounded Radon measures is extended to Čech-complete topological semigroups with separately continuous multiplication.

Let S be a separately continuous topological semigroup, $CB(S)$ the space of all bounded real-valued continuous functions on S and $M(S)$ the set of all bounded real-valued Radon measures on S . For μ in $M(S)$ and f in $CB(S)$, a result of Glicksberg [3] implies that the map $t \rightarrow \int f(st) d\mu(s)$ is continuous when restricted to the compacta of S . Hence it is ν -measurable for any ν in $M(S)$. Let $\psi(f) = \int [\int f(st) d\mu(s)] d\nu(t)$ for f in $CB(S)$. In this note it is shown that for a large class of semigroups, which includes the class of Čech-complete semigroups, ψ is represented by a unique bounded Radon measure. Denoting this measure by $\mu * \nu$, we have

$$\int f d\mu * \nu = \iint f(st) d\mu(s) d\nu(t) = \iint f(st) d\nu(t) d\mu(s)$$

for each f in $CB(S)$. This extends the convolution of a pair of bounded measures to a wide class of topological semigroups with separately continuous multiplication, including those considered in [1, 3, 6, 7].

All topological spaces are assumed to be completely regular Hausdorff. βX denotes the Stone-Čech compactification of the topological space X . f' is the unique extension of f in $CB(X)$ to $C(\beta X)$. For the set $A \subset X$, χ_A is the characteristic function of A . $\| \cdot \|$ denotes the uniform norm.

LEMMA 1. *Let S be a completely regular separately continuous topological semigroup. Let F, K be a pair of compacta of S , such that KF is measurable in βS with respect to every measure in $M(\beta S)$. Then for $\mu, \nu \in M(S)$ the functional, $\psi(f) = \int_F \int_K f(st) d\mu(s) d\nu(t)$ for f in $CB(S)$, is represented by a unique bounded Radon measure on S .*

PROOF. It is sufficient to prove the lemma for $\mu \geq 0$ and $\nu \geq 0$. Let γ' be a regular finite Borel measure on βS such that $\int_{\beta S} f' d\gamma' = \int_F \int_K f(st) d\mu(s) d\nu(t)$ for f in $CB(S)$. Let C be a nonvoid compact subset of $\beta S - KF$. Let $A = \{f: f \text{ in } CB(S), \chi_C \leq f' \leq 1\}$. Index the elements of A by $\{f_\lambda\}_{\lambda \in \Lambda}$ so that $\lambda > \eta$ iff $f_\lambda(x) \leq f_\eta(x)$ for each x in S . Since the minimum of any pair of elements of A is again in A , $\{f_\lambda\}_{\lambda \in \Lambda}$ is a (decreasing) net. For each λ in Λ , let $g_\lambda(t) = \int_K f_\lambda(st) d\mu(s)$. Since $\mu \geq 0$, $\{g_\lambda\}_{\lambda \in \Lambda}$ is a decreasing net. By Urysohn's lemma

Received by the editors July 24, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 43A05.

This work is supported by NSERC Grant A-7853.

©1988 American Mathematical Society
0002-9939/88 \$1.00 + \$.25 per page

$f_\lambda \downarrow 0$ on KF (note that the lemma is applied to the set $C, \{x\}$, where $x \in KF$, so that separate continuity suffices). Whence by Dini's theorem [5, p. 239] $f_\lambda \rightarrow 0$ uniformly on compacta of KF . Since $0 \leq f_\lambda \leq 1$ for each λ , $g_\lambda(t) \downarrow 0$ for each t in F . A similar application of Dini's theorem implies that $\int_F g_\lambda(t) d\nu(t) \rightarrow 0$. Therefore $0 \leq \gamma'(C) \leq \lim_\lambda \int_F \int_K f_\lambda(st) d\mu(s) d\nu(t) = 0$. Since KF is γ' -measurable in βS , $\gamma'(\beta S - KF) = 0$. Therefore $\gamma'(\beta S) = \gamma'(KF)$. Hence by [2, Corollary 4.4, p. 472] there is a unique bounded Radon measure γ on S such that $\int_{\beta S} f' d\gamma' = \int_S f d\gamma$ for each f in $CB(S)$.

REMARK. The above proof remains intact if KF is replaced by any subset A of S containing KF which is measurable with respect to every measure in $M(\beta S)$. In this case the assumption that KF is measurable is not needed.

THEOREM 2. *Let S be a completely regular separately continuous semigroup. Suppose for each pair of compacta F, K of S there is a subset A of S (depending on F, K) such that A is measurable with respect to every measure in $M(\beta S)$. Then the functional $\psi(f) = \iint f(st) d\mu(s) d\nu(t)$ for f in $CB(S)$ is represented by a unique bounded Radon measure on S for any μ and ν in $M(S)$.*

PROOF. Let $\mu, \nu \geq 0$. For $\varepsilon > 0$, let F, K be a pair of compact subsets of S such that $\mu(S - K) < \varepsilon$, $\nu(S - F) < \varepsilon$.

By Lemma 1, there is a bounded Radon measure γ on S such that

$$\int_F \int_K f(st) d\mu(s) d\nu(t) = \int f d\gamma$$

for each f in $CB(S)$. Let C be a compact subset of S such that $\gamma(S - C) < \varepsilon$. Then $|\int f d\gamma - \int_C f d\gamma| \leq \|f\|\varepsilon$. Now

$$\begin{aligned} \left| \psi(f) - \int_C f d\gamma \right| &\leq \left| \iint f(st) d\mu(s) d\nu(t) - \iint_K f(st) d\mu(s) d\nu(t) \right| \\ &\quad + \left| \iint_K f(st) d\mu(s) d\nu(t) - \int_F \int_K f(st) d\mu(s) d\nu(t) \right| \\ &\quad + \left| \int_F \int_K f(st) d\mu(s) d\nu(t) - \int_C f d\gamma \right| \\ &\leq \|f\|\mu(S - K)\nu(S) + \|f\|\mu(K)\nu(S - F) + \|f\|\varepsilon \\ &< \|f\|(\nu(S) + \mu(K) + 1)\varepsilon. \end{aligned}$$

By [4, Proposition 4.1 and 2, Lemma 4.5] it follows that there is a unique bounded Radon measure on S , which will be denoted by $\mu * \nu$, such that $\psi(f) = \int f d\mu * \nu$ for each f in $CB(S)$.

REMARKS. (a) For locally compact semigroups S with separately continuous multiplication and for μ, ν in $M(S)$, Glicksberg [3] proved that the map $t \rightarrow \int f(st) d\mu(s)$ is continuous on S for each f in $CB(S)$. This enabled him to obtain a bounded Radon measure $\mu * \nu$ such that $\int f d\mu * \nu = \iint f(st) d\mu(s) d\nu(t)$ for each continuous f vanishing at infinity. Later on Pym [6] and recently James C. S. Wong [7] proved that $\int f d\mu * \nu = \iint f(st) d\mu(s) d\nu(t)$ for each f in $CB(S)$. The assumption that S is locally compact is essential in their proofs. Since a locally compact S is open in βS , Theorem 2 implies all of the above.

(b) H. Dzinotyiweyi [1] obtained a similar result for any topological semigroup with jointly continuous multiplication. The joint continuity is essential in his proof. Theorem 2 provides a new proof for his result.

A topological space X is said to be Čech-complete if X is a G_δ in βX . Examples of Čech-complete spaces include locally compact spaces and complete metric spaces. The following now is immediate from Theorem 2.

COROLLARY 3. *Let S be a Čech-complete semigroup with separately continuous multiplication. Then for any μ, ν in $M(S)$, there is a unique measure $\mu * \nu$ in $M(S)$ such that $\int f d\mu * \nu = \iint f(st) d\mu(s) d\nu(t)$ for each f in $CB(S)$.*

Note that the Glicksberg result [3, Theorem 3.1] implies that $\iint f(st) d\mu(s) d\nu(t) = \iint f(st) d\nu(t) d\mu(s)$ for each f in $CB(S)$ for any topological semigroup S and any μ, ν in $M(S)$. (This follows from the middle inequalities in the proof of Theorem 2 and Glicksberg's result.)

ACKNOWLEDGEMENT. The author wishes to thank the referee for comments and suggestions.

REFERENCES

1. H. A. M. Dzinotyiweyi, *Algebras of measures on C -distinguished topological semigroups*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 323–336.
2. R. A. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.
3. J. Glicksberg, *Weak compactness and separate continuity*, Pacific J. Math. **11** (1961), 205–214.
4. R. A. Hirschfeld, *On measures in completely regular spaces*, Bull. Soc. Math. Belg. **24** (1972), 275–287.
5. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1965.
6. John S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc. (3) **15** (1965), 84–104.
7. James C. S. Wong, *Convolution and separate continuity*, Pacific J. Math. **75** (1978), 601–611.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1