OCCUPATION TIME AND THE LEBESGUE MEASURE OF THE RANGE FOR A LEVY PROCESS

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ABSTRACT. We consider a Levy process on the line that is transient and with nonpolar one point sets. For $a > 0$ let $N(a)$ be the total occupation time of $[0, a]$ and $R(a)$ the Lebesgue measure of the range of the process intersected with $[0, a]$. Whenever $[0, \infty)$ is a recurrent set we show $N(a)/EN(a) - R(a)/ER(a)$ converges in the mean square to 0 as $a \rightarrow \infty$. This in turn is used to derive limit laws for $R(a)/ER(a)$ from those for $N(a)/EN(a)$.

1. Introduction. Throughout this paper X_t will be a transient Levy process that is not a compound Poisson process starting at 0. Let $T_x = \inf\{t > 0: X_t = x\}$ be the hitting time of the singleton $\{x\}$. We will assume throughout that $\{0\}$ is not essentially polar, i.e. that $P_x(T_0 < \infty) > 0$ on a set of positive measure. Let $G(x, A) = \int_0^\infty P_x(X_t \in A) dt$. Under the above conditions $G(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure and there is a version of the density $g(x)$ having the following properties: (i) g is bounded. (ii) g is continuous except perhaps at 0 where it may have a jump. (iii) $g(0) = g(0+) \wedge g(0-)$. Let $C = [g(0+) \vee g(0-)]^{-1}$ and let $h(x) = P(T_x < \infty)$. Then $h(x) = Cg(x)$. The above facts can be found in [1].

For $a > 0$ let $N(a)$ be the total occupation time of $[0, a]$ and let $R(a)$ be the Lebesgue measure of the range of the process intersected with $[0, a]$. Observe that

$$(1.1) \quad ER(a) = \int_0^a h(x)dx = C \int_0^a g(x)dx = CEN(a).$$

Our goal in this paper is to investigate the asymptotic behavior of $R(a)$ and $N(a)$ as $a \rightarrow \infty$.

Pruitt and Taylor [6] seem to have been the first to investigate the asymptotic behavior of $R(a)$ as $a \rightarrow \infty$. They examined the asymmetric Cauchy processes. If the asymmetry parameter β is -1 (so the process has no positive jumps) then $[0, \infty)$ is a transient set and $R(a) \uparrow R(\infty)$ a.s. In all other cases $[0, \infty)$ is a recurrent set and $R(a) \uparrow \infty$ a.s. In this case Pruitt and Taylor [6] showed that

$$ER(a) \sim k_1 \frac{a}{\ln a}, \quad a \rightarrow \infty,$$

for a specified constant k_1 , and found the remarkable fact that

$$R(a) \ln a/a \xrightarrow{D} R$$

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where R or $R-1$ is geometrically distributed, according as $\beta < 0$ or $\beta > 0$. In [4] the asymptotic behavior of $N(a)$ was determined for the asymmetric Cauchy processes. If $\beta = -1$, $N(a) \uparrow N(\infty) < \infty$ a.s. For $\beta > -1$, $N(a) \ln a/a \xrightarrow{D} (1/C)R$. Hence for $\beta > -1$ we have the curious fact that $N(a)/EN(a)$ and $R(a)/ER(a)$ have the same limiting distribution.

Let $\tilde{X}_t$ be a stable process with exponent α and asymmetry parameter β . For $b \neq 0$, the process $X_t = \tilde{X}_t + bt$ is the stable process with drift b . Suppose $\alpha < 1$. Then as for the asymmetric Cauchy processes, $[0, \infty)$ is a transient set if $\beta = -1$. In all other cases $[0, \infty)$ is a recurrent set. It was shown in [4] that for $\beta > -1$, $EN(a) \sim k_2 a^\alpha$, $a \rightarrow \infty$, for a specified numerical constant k_2 , and that $N(a)/EN(a) \xrightarrow{D} N$, $a \rightarrow \infty$, where N is the total occupation time of $[0, 1]$ by the process $\tilde{X}_t$.

One of the motivations for the present paper was to find the limiting distribution for $R(a)/ER(a)$ for the stable processes with drift, exponent $\alpha < 1$, and $\beta > -1$. Another motivation was to understand why the limiting distributions of $R(a)/ER(a)$ and $N(a)/EN(a)$ were the same for the asymmetric Cauchy processes with $\beta > -1$. Remarkably, the following holds.

THEOREM 1. *Suppose $[0, \infty)$ is a recurrent set. Then*

$$(1.2) \quad E \left[\frac{R(a)}{ER(a)} - \frac{N(a)}{EN(a)} \right]^2 \rightarrow 0, \quad a \rightarrow \infty.$$

If $[0, \infty)$ is a transient set $R(a) \uparrow R(\infty)$ and $N(a) \uparrow N(\infty)$, $N(\infty)$ and $R(\infty)$ are $< \infty$ a.s., and generally there is no connection between the distributions of $R(\infty)$ and $N(\infty)$.

Theorem 1 shows that when $[0, \infty)$ is a recurrent set then $N(a)/EN(a)$ and $R(a)/ER(a)$ are asymptotically equal. This not only explains the equality of the limiting distributions for the asymmetric Cauchy processes with $\beta > -1$, but also yields a powerful tool for finding the limiting distributions for $R(a)/ER(a)$.

In general, it is usually quite difficult to find the limiting distribution of $R(a)/ER(a)$ by direct methods, even when considerable information is available about the asymptotic behavior of $g(x)$. In contrast, it is usually a routine application of the method of moments to determine the limiting distribution of $N(a)/EN(a)$ (provided it exists). See e.g. [4]. We will illustrate the use of Theorem 1 by finding the limiting distributions of $R(a)/ER(a)$ for the stable processes with drift and exponent $\alpha < 1$, and by rederiving the Pruitt-Taylor result.

Let X_t be a stable process with drift and exponent $\alpha < 1$. Assume $\beta > -1$. Let

$$k_2 = \left[1 + \beta^2 \tan \left(\frac{\pi\alpha}{2} \right) \right]^{-1} \Gamma \left(\frac{1-\alpha}{\pi} \right) \sin \left(\frac{\pi\alpha}{2} \right) \alpha^{-1} (1 + \beta).$$

Then it was shown in [4] that $EN(a) \sim k_2 a^\alpha$, $a \rightarrow \infty$, and, as mentioned above, $N(a)/k_2 a^\alpha \xrightarrow{D} N$. Using these facts, (1.1), and Theorem 1, we can conclude the following.

THEOREM 2. *For a stable process with drift and exponent $\alpha < 1$ and $\beta > -1$*

$$R(a)/Ck_2 a^\alpha \xrightarrow{D} N.$$

Note: An explicit formula for C can be found in [4].

As a final illustration of the use of Theorem 1 let X_t be an asymmetric Cauchy process. Let Z be a geometrically distributed random variable with parameter $\rho = (1 - \beta)/(1 + \beta)$ if $\beta > 0$ and with parameter $(1 + \beta)/(1 - \beta)$ if $\beta < 0$. Again, by an easy application of the method of moments, it was shown in [4] that

$$\frac{N(a)}{a/\ln a} \xrightarrow{D} \frac{1}{C} (Z + 1) \quad \text{if } \beta > 0$$

and

$$\frac{N(a)}{a/\ln a} \xrightarrow{D} \frac{Z}{C} \quad \text{if } \beta < 0.$$

Using Theorem 1 we can deduce the result due to Pruitt and Taylor [6] that

$$\frac{R(a)}{a/\ln a} \xrightarrow{D} (Z + 1) \quad \text{if } \beta > 0 \quad \text{and} \quad \frac{R(a)}{a/\ln a} \xrightarrow{D} Z \quad \text{if } \beta < 0.$$

The proof given here via Theorem 1 seems to be substantially simpler than those given by Pruitt and Taylor [5, 6].

To prove Theorem 1 we will need the renewal theorem for $g(x)$. Recall that a process is called type I transient except if it is transient with finite mean b . It is then called type II.

THEOREM 3. *Assume the process is type I. Then $\lim_{|x| \rightarrow \infty} g(x) = 0$. Assume the process is type II transient and $b > 0$. Then $\lim_{x \rightarrow -\infty} g(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 1/b$.*

Both the stable processes with drift and exponent $\alpha < 1$ and the asymmetric Cauchy processes have infinite mean. It is only for Levy processes with infinite mean that there is any real problem with the asymptotic behavior of $N(a)$ and $R(a)$. If the process X_t has finite mean things are quite simple. Suppose $EX_t = bt$. If $b < 0$ then $[0, \infty)$ is a transient set since $X_t \rightarrow -\infty$ a.s. On the other hand if $b > 0$, $X_t \rightarrow \infty$ a.s. and $[0, \infty)$ is recurrent. Theorem 3 shows that for $b > 0$, $g(x) \rightarrow 1/b$ as $x \rightarrow \infty$ and $g(x) \rightarrow 0$ as $x \rightarrow -\infty$. Using this fact an easy computation shows that $E_0(N(a)/a - 1/b)^2 \rightarrow 0$. Thus by Theorem 1 $N(a)/EN(a)$ and $R(a)/ER(a)$ converge in L_2 to 1. This can be strengthened to a.s. convergence. That $N(a)/EN(a) \rightarrow 1$ a.s. is a routine application of the strong law of large numbers as we shall see. To establish that $R(a)/ER(a) \rightarrow 1$ a.s. we will need to extend a result due to Kesten, Spitzer, and Whitman, for Brownian motion on R^d to the Levy processes considered here. Since this fact actually holds in maximum generality we will take this opportunity to place the fact in the literature.

Let $\mathcal{G}$ be a second countable locally compact Abelian group and let X_t be an I.D. process on $\mathcal{G}$ such that the smallest closed subgroup of $\mathcal{G}$ containing all of the possible points of X_t (see p. 158 of [3] for the definition of a possible point) is $\mathcal{G}$, and assume X_t is transient. Let $C_B(t)$ be the Haar measure of the set $\{x: x \in X_s + B \text{ for some } s \leq t\}$ where B is a compact set.

For Brownian motion on R^d , $d \geq 3$, Kesten, Spitzer, and Whitman showed that $\lim_{t \rightarrow \infty} C_B(t)/t = C(B)$ with probability one, where $C(B)$ is the capacity of B . This result was never published, but an outline of the proof can be found in [2, p. 252]. The full proof is in W. W. Whitman's 1964 Cornell University thesis.

Examination of the proof reveals that in fact the process X_t is Brownian motion if used in only two ways. (1) X_t is an I. D. process. (2) Let $T_B = \inf\{t > 0: X_t \in B\}$. Then

$$(1.3) \quad \frac{1}{t} \int P_x(T_B \leq t) dx \rightarrow C(B), \quad t \rightarrow \infty,$$

and

$$(1.4) \quad \int P_x(T_B \leq t, X_s \notin B \text{ for all } s > t) dx = tC(B).$$

In [3] a capacity theory was developed for all transient I. D. processes on $\mathcal{G}$ and the analogs of (1.3) and (1.4) were shown to be valid. ((1.3) is (11.3) of [3] and (1.4) is (11.14) of [3]). Consequently, the same proof given for the Brownian motion case shows the following holds.

THEOREM 4. *Let $C_B(t)$ be as above. Then $\lim_{t \rightarrow \infty} C_B(t)/t = C(B)$ with probability one.*

We will use Theorem 4 for $B = \{0\}$ to show the following.

THEOREM 5. *Let X_t be a type II transient process on R with mean $b > 0$. Then $N(a)/a \rightarrow 1/b$ with probability one and $R(a)/a \rightarrow C/b$ with probability one as $a \rightarrow \infty$. Also $N(a)/a$ and $R(a)/Ca$ converge in L_2 to $1/b$.*

Theorems 1 and 5 apply with obvious modifications to the occupation time $N(a_1, a_2]$ of $(a_1, a_2]$ and the Lebesgue measure of the range intersected with $(a_1, a_2]$ as $a_1 \rightarrow -\infty$, $a_2 \rightarrow \infty$. We omit the details.

More generally, let $\mathcal{G}$ be a second countable locally compact Abelian group and X_t a Levy process on $\mathcal{G}$ as described above. Assume X_t is transient and that $\{0\}$ is not essentially polar. Let $A_1 \subseteq A_2 \subseteq \cdots$, $\bigcup_n A_n = A$. Assume A_i are relatively compact and that A is recurrent. Let $N(A_i)$ be the occupation time of A_i and $R(A_i)$ the Haar measure of the range of the process intersected with A_i . Then the same arguments show that

$$E \left(\frac{N(A_i)}{EN(A_i)} - \frac{R(A_i)}{ER(A_i)} \right)^2 \rightarrow 0.$$

We omit the details.

2. Proofs.

PROOF OF THEOREM 3. We will establish the assertions of Theorem 3 for a type II process with mean $b > 0$. The same kind of argument can be used to establish the assertions of the theorem for a type I process. As usual, let C be the capacity of a point. Equation (7.6) of [3] shows $h(x+y) \geq h(x)h(y)$ or equivalently,

$$(2.1) \quad g(x+y) \geq Cg(x)g(y).$$

Let $a > 0$ and $h > 0$. Then (2.1) shows

$$(2.2) \quad \int_a^{a+h} g(y)dy \geq Cg(a) \int_0^h g(x)dx.$$

Using the renewal theorem (Theorem 10.1 of [1]) we find from (2.2) that

$$(2.3) \quad \overline{\lim}_{a \rightarrow \infty} g(a) \leq \frac{1}{b} \left[\frac{C}{h} \int_0^h g(x) dx \right]^{-1}.$$

In all cases when $b > 0$, g is left continuous at 0 and $g(0+) = C^{-1}$. (See [1].) Thus letting $h \downarrow 0$ in (2.3) we find

$$(2.4) \quad \overline{\lim}_{a \rightarrow \infty} g(a) \leq \frac{1}{b}.$$

Taking $x + y = a$ in (2.1) we find

$$g(a) \geq Cg(x)g(a-x).$$

Thus

$$(2.5) \quad hg(a) \geq C \int_{a-h}^a g(x)g(a-x) dx.$$

Since $g(0+) = C^{-1}$ we can, for any given $\varepsilon > 0$, find $h > 0$ such that $Cg(y) \geq 1 - \varepsilon$ if $0 < y < h$. Thus for such h

$$g(a) \geq \frac{1 - \varepsilon}{h} \int_{a-h}^a g(x) dx.$$

Using the renewal theorem again we find

$$(2.6) \quad \underline{\lim}_{a \rightarrow \infty} g(a) \geq \frac{1 - \varepsilon}{b}.$$

It follows from (2.4) and (2.6) that $\lim_{a \rightarrow \infty} g(a) = 1/b$. The renewal theorem also shows

$$\lim_{a \rightarrow -\infty} \int_a^{a+h} g(x) dx = \lim_{a \rightarrow -\infty} \int_{a-h}^a g(x) dx = 0.$$

Using this fact the same kind of argument shows $\lim_{a \rightarrow -\infty} g(x) = 0$.

PROOF OF THEOREM 1. Observe that

$$(2.7) \quad EN(a)^2 = 2 \int_0^a \int_0^a g(x)g(y-x) dx dy.$$

Now

$$\begin{aligned} ER(a)^2/C^2 &= \frac{1}{C^2} \int_0^a \int_0^a P(T_x < \infty, T_y < \infty) dx dy \\ &= \frac{1}{C^2} \left[\int_0^a \int_0^a P(T_x < T_y < \infty) + P(T_y < T_x < \infty) \right] dx dy \\ &= \frac{1}{C^2} \int_0^a \int_0^a [P(T_x < T_y)P_x(T_y < \infty) + P(T_y < T_x)P_y(T_x < \infty)] dx dy \\ &\leq \int_0^a \int_0^a [g(x)g(y-x) + g(y)g(x-y)] dx dy. \end{aligned}$$

Thus

$$(2.8) \quad ER(a)^2/C^2 \leq EN(a)^2.$$

Now

$$R(a)N(a) = \int_0^\infty dt \int_0^a dx \, 1_{[T_x < \infty]} 1_{[0, a]}(X_t).$$

Thus

$$(2.9) \quad E[R(a)N(a)] = \int_0^\infty dt \int_0^a dx \, P(T_x < \infty, 0 \leq X_t \leq a).$$

But

$$(2.10) \quad P(T_x < \infty, 0 < X_t \leq a) = \int_0^t P(T_x \in ds) P_x(0 \leq X_{t-s} \leq a) \\ + P(t < T_x < \infty, 0 \leq X_t \leq a).$$

Integrating (2.10) over t we find

$$(2.11) \quad \int_0^\infty P(T_x < \infty, 0 \leq X_t \leq a) dt \\ = P(T_x < \infty) \int_0^a g(y-x) dy + \int_0^\infty P(t < T_x < \infty, 0 \leq X_t \leq a) dt.$$

Now

$$\int_0^\infty P(t < T_x < \infty, X_t \in [0, a]) dt \\ = \int_0^\infty \int_0^a P(T_x > t, X_t \in dy) P_y(T_x < \infty) dt.$$

Let $G_{\{x\}}(0, A) = \int_0^\infty P(T_x > t, X_t \in A) dt$ and let $G(x, A) = \int_0^\infty P_x(X_t \in A) dt$. The first passage equation shows

$$G_{\{x\}}(0, A) = G(0, A) - P(T_x < \infty)G(x, A).$$

Since $G(x, dy) = g(y-x) dy$ it follows that

$$\int_0^a \int_0^\infty P(T_x > t, X_t \in dy) P_y(T_x < \infty) dt = \int_0^a G_{\{x\}}(0, dy) P_y(T_x < \infty) \\ = \int_0^a g(y) P_y(T_x < \infty) dy - P(T_x < \infty) \int_0^a g(y-x) P_y(T_x < \infty) dy \\ = C \int_0^a [g(y)g(x-y) dy - C^2 g(x)g(y-x)g(x-y)] dy.$$

Using (2.11) and (2.9) we find

$$(2.12) \quad E[R(a)N(a)] = C \int_0^a \int_0^a g(x)g(y-x) dx dy \\ + C \int_0^a \int_0^a g(y)g(x-y) dx dy \\ - C^2 \int_0^a \int_0^a g(x)g(y-x)g(x-y) dx dy.$$

Using (2.7), (2.8), and (2.12) we find

$$(2.13) \quad E \left[\frac{R(a)}{C} - N(a) \right]^2 \leq 2C \int_0^a \int_0^a g(x)g(y-x)g(x-y) dx dy.$$

Now (1.2) will hold provided we can show

$$(2.14) \quad \lim_{a \rightarrow \infty} (EN(a))^{-2} \int_0^a \int_0^a g(x)g(y-x)g(x-y) dx dy = 0.$$

The assumption that $[0, \infty)$ is a recurrent set implies that the process is either type I or type II with mean $b > 0$. In either case $\lim_{x \rightarrow -\infty} g(x) = 0$. Thus for given $\varepsilon > 0$ we can find $\delta < \infty$ such that $g(-z) \leq \varepsilon$ for all $z \geq \delta$. Let $\psi(x) = \int_0^x g(z)g(-z)dz$ and $M = \sup_x g(x) < \infty$. Now

$$\begin{aligned} \int_0^a \int_0^a g(x)g(y-x)g(x-y) dx dy &= \int_0^a g(a-x)\psi(x) dx + \int_0^a g(x)\psi(x) dx \\ &= \int_0^\delta [g(a-x) + g(x)]\psi(x) dx + \int_\delta^a [g(a-x) + g(x)]\psi(x) dx \\ &\leq M^3\delta^2 + 2\varepsilon \left(\int_0^a g(x) dx \right)^2 + 2M^2\delta \int_0^a g(x) dx. \end{aligned}$$

Thus

$$\overline{\lim}_{a \rightarrow \infty} (EN(a))^{-2} \int_0^a \int_0^a g(x)g(y-x)g(x-y) dx dy \leq 2\varepsilon.$$

Hence (2.14) holds.

PROOF OF THEOREM 5. The strong law of large number shows $X_t/t \rightarrow b$ as $t \rightarrow \infty$ with probability one. Fix $\varepsilon > 0$. Then $\exists t_0(\omega)$ such that for $t \geq t_0(\omega)$, $(b - \varepsilon)t \leq X_t(\omega) \leq (b + \varepsilon)t$ for all ω except perhaps for ω in a set of probability 0. Assume $b - \varepsilon > 0$ and $a > (b + \varepsilon)t_0$.

$$N(a) \geq \int_{t_0}^{a/b+\varepsilon} 1_{[0,a]}(X_t) dt = \frac{a}{b+\varepsilon} - t_0$$

and

$$N(a) \leq \int_0^{a/b-\varepsilon} 1_{[0,a]}(X_t) dt \leq \frac{a}{b-\varepsilon}$$

except for ω in the exceptional set. Hence

$$\frac{1}{b+\varepsilon} \leq \lim_{a \rightarrow \infty} \frac{N(a)}{a} \leq \overline{\lim}_{a \rightarrow \infty} \frac{N(a)}{a} \leq \frac{1}{b-\varepsilon}.$$

Thus $N(a)/a \rightarrow 1/b$ with probability one. Theorem 3 shows $EN(a) = \int_0^a g(x)dx \sim a/b$ as $a \rightarrow \infty$.

Let R_t be the Lebesgue measure of $\{x: x = X_s \text{ for some } s \leq t\}$. Then for ω not in the exceptional set

$$R(a) \leq R_t$$

for all $t \geq (b - \varepsilon) \wedge t_0$. Assume $a/b - \varepsilon > t_0$. Then $R(a) \leq R_{a/b-\varepsilon}$. For $B = \{0\}$, $\{X_s + B, s \leq t\} = \{X_s, s \leq t\}$. Thus by Theorem 4

$$\lim_{t \rightarrow \infty} R_t/t = C$$

with probability one. Hence for ω not in some set of probability 0.

$$\overline{\lim}_{a \rightarrow \infty} \frac{R(a)}{a} \leq \lim_{a \rightarrow \infty} \frac{R_{a/b-\varepsilon}}{a} = \frac{C}{b-\varepsilon}.$$

Similarly

$$\lim_{a \rightarrow \infty} \frac{R(a)}{a} \geq \lim_{a \rightarrow \infty} \frac{R_{a/b+\varepsilon} - R_{t_0}}{a} = \frac{C}{b+\varepsilon}$$

except for ω in the exceptional set. Thus $\lim_{a \rightarrow \infty} R(a)/a = C/b$ with probability one. Since $0 \leq R(a)/a \leq 1$, $ER(a)^2/(Ca)^2 \rightarrow 1/b^2$. Thus $R(a)/Ca \xrightarrow{L_2} 1/b$. Theorem 1 now shows that $N(a)/a \xrightarrow{L_2} 1/b$.

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