AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF A RANDOM POLYNOMIAL

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ABSTRACT. Let ν_n be the expected number of real zeros of a polynomial of degree n whose coefficients are independent random variables, normally distributed with mean 0 and variance 1. We find an asymptotic expansion for ν_n of the form

$$\nu_n = \frac{2}{\pi} \log(n+1) + \sum_{p=0}^{\infty} A_p (n+1)^{-p},$$

in which $A_0 = 0.625735818$, $A_1 = 0$, $A_2 = -0.24261274$, $A_3 = 0$, $A_4 = -0.08794067$, $A_5 = 0$. The numerical values of ν_n calculated from this expansion, using only the first four, or six, coefficients, agree with previously tabulated seven decimal place values ($1 \leq n \leq 100$) with an error of at most 10^{-7} when $n \geq 30$, or $n \geq 8$.

1. Introduction. Suppose that the coefficients of the polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ are independent, normally distributed random variables with mean 0 and variance 1. Kac [1] has shown that the expected number of real zeros of $f(z)$ is

$$(1) \quad \nu_n = \frac{4}{\pi} \int_0^1 \frac{\{1 - h_n^2(x)\}^{1/2}}{1 - x^2} dx,$$

in which $h_n(x) = (n+1)x^n(1-x^2)/(1-x^{2n+2})$. Kac [1] also showed that $\nu_n \sim (2/\pi) \log n$ for large n . Jamrom [2, 3] improved this result and showed that

$$(2) \quad \lim_{n \rightarrow \infty} \{\nu_n - (2/\pi) \log(n+1)\} = A_0,$$

$$(3) \quad A_0 = \frac{2}{\pi} \left\{ \log 2 + \int_0^1 (1 - t^2 \operatorname{csch}^2 t)^{1/2} t^{-1} dt - \int_1^\infty \{1 - (1 - t^2 \operatorname{csch}^2 t)^{1/2}\} t^{-1} dt \right\}.$$

Wang [4] also derived (2); his value of A_0 is

$$\frac{8}{\pi} \int_0^1 (1 - y^2)^{-1} \left\{ 1 + \frac{(1 - y^2 - 2y \log y)^{1/2}}{(1 - y^2 + 2y \log y)^{1/2}} \right\}^{-1} dy.$$

(The substitution $y = e^{-t}$ and some simple manipulations show that the two values of A_0 are the same.)

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Several authors [1, 4–8] have found upper and/or lower bounds for ν_n , all of which have been superseded, or anticipated, by the result of Yu [9] that $\nu_n - (2/\pi) \log(n+1)$ is an increasing sequence, so that

$$(4) \quad 0.558728799 < \nu_1 - (2/\pi) \log 2 \leq \nu_n - (2/\pi) \log(n+1) < A_0.$$

In this paper we will prove the following result.

THEOREM. *There is an asymptotic expansion of the form*

$$(5) \quad \nu_n \sim (2/\pi) \log(n+1) + \sum_{p=0}^{\infty} A_p (n+1)^{-p},$$

in which A_0 is defined by (3), $A_1 = A_3 = A_5 = 0$, and

$$A_2 = -\frac{1}{3\pi} \int_0^{\infty} \{(1 - t^2 \operatorname{csch}^2 t)^{-1/2} - 1\} t \, dt,$$

$$A_4 = -\frac{1}{180\pi} \int_0^{\infty} \{7 - 12(1 - t^2 \operatorname{csch}^2 t)^{-1/2} + 5(1 - t^2 \operatorname{csch}^2 t)^{-3/2}\} t^3 \, dt.$$

The discovery that $A_1 = A_3 = A_5 = 0$ was unexpected. (We have been unable to decide the irresistible conjecture that $A_p = 0$ for all odd p .) We note that $A_0 = 0.625735818$, $A_2 = -0.24261274$, $A_4 = -0.08794067$. Our numerical value for A_0 does not agree with the value 0.6312 reported by Wang [4].

2. Preliminaries. In this section we perform certain algebraic manipulations useful for the subsequent analysis. The variable x is a number on the interval $(0, 1)$, and n and $N-1$ are positive integers. It is convenient to assume that $n \geq 8$.

We define δ, a, t, w and μ so that $\delta = 1 - (n+1)^{-1/2}$, $a = (n+1)^{1/2}$, $x = 1 - t/(n+1)$, $x^n = e^{w-t}$, $x^{2n+2} = e^{-\mu-2t}$. Then $\delta \geq 2/3$, $a \geq 3$, $0 \leq t \leq a$ when $\delta \leq x \leq 1$, $w = t + n \log x$, $\mu = -2t - (2n+2) \log x$. It is now easy to verify the results stated in the following lemma.

LEMMA 1. *If B, C, D and E are defined so that*

$$B = e^{2w} - 1, \quad C = t/2(n+1), \quad D = (1 - e^{-\mu})/(e^{2t} - 1),$$

$$E = \frac{(2D - B + 2C + D^2 - C^2 + 2BC - BC^2)t^2 \operatorname{csch}^2 t}{(1 + D)^2(1 - t^2 \operatorname{csch}^2 t)},$$

then

$$h_n^2(x) = 4t^2 e^{-2t} (1+B)(1-C)^2 (1 - e^{-2t})^{-2} (1+D)^{-2},$$

$$1 - h_n^2(x) = (1 - t^2 \operatorname{csch}^2 t)(1+E).$$

According to Taylor's Theorem, there is a number θ_1 , depending on x and N , such that $0 < \theta_1 < 1$ and

$$\log x = - \sum_{p=1}^N p^{-1} t^p (n+1)^{-p} - (N+1)^{-1} t^{N+1} (n+1 - \theta_1 t)^{-N-1}.$$

Because $n+1 \geq n+1 - \theta_1 t \geq n+1 - a \geq 2(n+1)/3$ when $0 \leq t \leq a$ and $n \geq 8$, the number v_N defined as $\{(n+1)/(n+1 - \theta_1 t)\}^{N+1}$ is such that $1 \leq v_N \leq (3/2)^{N+1}$. It is now easy to see that the following lemma is true.

LEMMA 2. If $\psi_{pN}(t)$ and $\phi_{pN}(t)$ are defined so that

$$\begin{aligned}\psi_{pN} &= 2/(p+1), & \phi_{pN} &= (p+1-pt)/p(p+1) \quad \text{when } p = 1, 2, \dots, N-1, \\ \psi_{NN} &= 2v_N/(N+1), & \phi_{NN} &= N^{-1} - nv_N t/(N+1)(n+1),\end{aligned}$$

then

$$\mu = t \sum_{p=1}^N \psi_{pN}(t) t^p (n+1)^{-p}, \quad w = \sum_{p=1}^N \phi_{pN}(t) t^p (n+1)^{-p}.$$

We furnish upper bounds for w and e^{2w} in the following lemma.

LEMMA 3. If $0 \leq x \leq 1$ and $n \geq 8$, then $w \leq 1 - 8 \log(9/8) < 0.05774$, and $e^{2w} \leq (8/9)^{16} e^2 < 1.1225$.

If we consider $w = (n+1)(1-x) + n \log x$ as a function of x on $(0, 1)$, we see that $dw/dx = -n-1+nx^{-1}$, $d^2w/dx^2 = -nx^{-2} < 0$. It follows that w attains its absolute maximum $w_{\max}$ when $x = n/(n+1)$, and that

$$w_{\max} = 1 + n \log\{1 - (n+1)^{-1}\} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)(n+1)^p}.$$

Hence $w_{\max}$ is a decreasing function of n . This remark and some arithmetic suffice to prove the lemma.

We can now find a lower bound for E .

LEMMA 4. If $\delta \leq x \leq 1$ and $n \geq 8$, then $E \geq -0.6508$.

It follows from Lemma 2 when $\delta \leq x \leq 1$ and $n \geq 8$ that

$$\begin{aligned}0 \leq \frac{t^2}{n+1} \leq \mu &= \frac{t^2}{n+1} \left\{ 1 + \frac{2v_2 t}{3(n+1)} \right\} \leq \frac{t^2}{n+1} \left\{ 1 + \frac{9t}{4(n+1)} \right\}, \\ 2D &\geq (2\mu - \mu^2)/(e^{2t} - 1).\end{aligned}$$

Moreover, $B = e^{2w} - 1 = 2w + 2w^2 \exp(2\theta_2 w)$ for some θ_2 such that $0 < \theta_2 < 1$. It then follows from Lemma 3 that

$$2w \leq B \leq 2w + 2w^2 \exp\{\max(2w, 0)\} \leq 2w + 2.245w^2.$$

We infer from the definitions of w and t that

$$\begin{aligned}(n+1)^{-1} &= 1 - n(n+1)^{-1} \geq \frac{w}{t} = 1 - \sum_{p=1}^{\infty} np^{-1} t^{p-1} (n+1)^{-p} \\ &\geq 1 - \sum_{p=1}^{\infty} np^{-1} a^{-p-1} = -\frac{1}{2a} + \sum_{p=2}^{\infty} \frac{2}{(p^2-1)a^p}.\end{aligned}$$

Hence $w \leq t/(n+1)$, $-1/2a \leq w/t \leq 1/2a$, $w^2 \leq t^2/4(n+1)$.

If $R = 2D - B + 2C + D^2 - C^2 + BC(2 - C)$, we now see that

$$\begin{aligned}
 -R \leq & -\frac{2\mu - \mu^2}{e^{2t} - 1} + (2w + 2.245w^2) - \frac{t}{n+1} - 0 \\
 & + \frac{t^2}{4(n+1)^2} - \frac{2wt}{n+1} \left\{ 1 - \frac{t}{4(n+1)} \right\} \leq -\frac{2t^2}{(n+1)(e^{2t} - 1)} \\
 & + \frac{t^4}{(n+1)^2(e^{2t} - 1)} \left\{ 1 + \frac{9t}{2(n+1)} \right\} + \frac{81t^2}{16(n+1)^2} \\
 & + \frac{2t}{n+1} + \frac{0.56125t^2}{n+1} - \frac{t}{n+1} + \frac{t^2}{4(n+1)^2} \\
 & + \frac{t^2}{(n+1)^{3/2}} = t^2(n+1)^{-1} \{ P_0(t) + (n+1)^{-1/2} + P_1(t)(n+1)^{-1} \\
 & \quad + P_2(t)(n+1)^{-2} + P_3(t)(n+1)^{-3} \},
 \end{aligned}$$

in which

$$\begin{aligned}
 P_0(t) &= t^{-1} - 2(e^{2t} - 1)^{-1} + 0.56125, \quad P_1(t) = t^2(e^{2t} - 1)^{-1} + \frac{1}{4}, \\
 P_2(t) &= 4.5t^3(e^{2t} - 1)^{-1}, \quad \text{and} \quad P_3(t) = 5.0625t^4(e^{2t} - 1)^{-1}.
 \end{aligned}$$

It is a straightforward task to estimate the maxima of the functions $P_l(t)$. We find that $P_0(t) \leq 1.56125$, $P_1(t) \leq 0.4120$, $P_2(t) \leq 0.7996$, $P_3(t) \leq 1.5124$. Because $n \geq 8$ and $D \geq 0$, it follows that $-R \leq 1.9524t^2(n+1)^{-1}$, $-E \leq 1.9524\chi(t)(n+1)^{-1}$, in which

$$\chi(t) = t^4(1 - t^2 \operatorname{csch}^2 t)^{-1} \operatorname{csch}^2 t = t^4/(\sinh^2 t - t^2).$$

The validity of Lemma 4 is now a consequence of the inequality, $\sinh^2 t - t^2 \geq t^4/3$.

There is a number θ_3 such that $0 < \theta_3 < 1$ and

$$B = e^{2w} - 1 = \sum_{k=1}^N \frac{2^k b_{kN} w^k}{k!},$$

in which $b_{kN} = 1$ when $k = 1, 2, \dots, N-1$, and $b_{NN} = \exp(2\theta_3 w) < 1.1225$. It now follows from Lemma 2 that

$$B = \sum_{p=1}^{N^2} \beta_{pN}(t) t^p (n+1)^{-p},$$

in which β_{pN} is a polynomial of degree p in t , whose coefficients are independent of both n and N , when $1 \leq p \leq N-1$, and β_{pN} is bounded when $0 \leq t \leq a$ by a polynomial of degree N in t , whose coefficients are independent of n , when $N \leq p \leq N^2$. Therefore,

$$B = \sum_{p=1}^N B_{pN}(t) t^p (n+1)^{-p},$$

in which $B_{pN} = \beta_{pN}$ when $1 \leq p \leq N-1$, and

$$B_{NN} = \sum_{p=N}^{N^2} \beta_{pN} t^{p-N} (n+1)^{-p+N},$$

so that B_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree N in t whose coefficients are independent of n . In particular,

$$\begin{aligned} B_{1N} &= 2 - t, & B_{2N} &= \frac{1}{6}(18 - 16t + 3t^2), & B_{3N} &= \frac{1}{6}(24 - 29t + 10t^2 - t^3), \\ B_{4N} &= \frac{1}{360}(1800 - 2664t + 1280t^2 - 240t^3 + 15t^4), \\ B_{5N} &= \frac{1}{360}(2160 - 3708t + 2224t^2 - 590t^3 + 70t^4 - 3t^5) \end{aligned}$$

when $N \geq 6$.

In a similar manner we find that

$$D = \omega(t) \sum_{p=1}^N D_{pN}(t) t^p (n+1)^{-p},$$

in which $\omega(t) = t/(e^{2t} - 1) = t(\coth t - 1)/2$, and D_{pN} is a polynomial of degree $p - 1$ in t , whose coefficients are independent of n and N , when $1 \leq p \leq N - 1$, and D_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree $N - 1$ in t whose coefficients are independent of n . In particular,

$$\begin{aligned} D_{1N} &= 1, & D_{2N} &= \frac{1}{6}(4 - 3t), & D_{3N} &= \frac{1}{6}(3 - 4t + t^2), \\ D_{4N} &= \frac{1}{360}(144 - 260t + 120t^2 - 15t^3), \\ D_{5N} &= \frac{1}{360}(120 - 264t + 170t^2 - 40t^3 + 3t^4), \end{aligned}$$

when $N \geq 6$. It follows that

$$R = \sum_{p=1}^N R_{pN}(t) t^p (n+1)^{-p},$$

in which R_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree p in t when $1 \leq p \leq N - 1$. Also, R_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree N in t whose coefficients are independent of n . In particular,

$$\begin{aligned} R_{1N} &= 2\omega - 1 + t = t(\coth t - t^{-1}), & R_{2N} &= \omega^2 + \frac{1}{3}(4 - 3t)\omega - \frac{1}{12}(15 - 20t + 6t^2), \\ R_{3N} &= \frac{1}{3}(4 - 3t)\omega^2 + \frac{1}{3}(3 - 4t + t^2)\omega - \frac{1}{12}(18 - 29t + 14t^2 - 2t^3), \\ R_{4N} &= \frac{1}{36}(52 - 72t + 21t^2)\omega^2 + \frac{1}{180}(144 - 260t + 120t^2 - 15t^3)\omega \\ &\quad - \frac{1}{360}(630 - 1164t + 725t^2 - 180t^3 + 15t^4), \\ R_{5N} &= \frac{1}{180}(264 - 510t + 280t^2 - 45t^3)\omega^3 + \frac{1}{180}(120 - 264t + 170t^2 - 40t^3 + 3t^4)\omega \\ &\quad - \frac{1}{360}(720 - 1479t + 1094t^2 - 365t^3 + 55t^4 - 3t^5), \end{aligned}$$

when $N \geq 6$. The same sort of reasoning shows that

$$(1 + D)^{-2} = 1 + \omega(t) \sum_{p=1}^{N-1} u_{pN}(t) t^p (n+1)^{-p},$$

in which u_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree $p - 1$ in t when $1 \leq p \leq N - 2$, and $u_{N-1,N}$ is bounded when $0 \leq t \leq a$ by a polynomial of degree $N - 2$ in t whose coefficients are independent of n . (It is necessary to observe that $D \geq 0$.) In particular,

$$\begin{aligned} u_{1N} &= -2, & u_{2N} &= 3\omega - \frac{1}{3}(4 - 3t), & u_{3N} &= -4\omega^2 + (4 - 3t)\omega - \frac{1}{3}(3 - 4t + t^2), \\ u_{4N} &= 5\omega^3 - (8 - 6t)\omega^2 + \frac{1}{12}(52 - 72t + 21t^2)\omega - \frac{1}{180}(144 - 260t + 120t^2 - 15t^3), \end{aligned}$$

when $N \geq 6$.

We next observe that

$$(6) \quad E = t^{-2} \chi(t) R(1 + D)^{-2} = \chi(t) \sum_{p=1}^N E_{pN}(t)(n+1)^{-p},$$

in which E_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree $2p - 2$ in t when $1 \leq p \leq N - 1$. (The bound is not obvious when $p = 1$; it is true because $E_{1N} = t^{-1} R_{1N} = \coth t - t^{-1}$.) Moreover, E_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree $2N - 2$ in t whose coefficients are independent of n . In particular,

$$\begin{aligned} E_{2N} &= -3\omega^2 + \frac{1}{3}(10 - 9t)\omega - \frac{1}{12}(15 - 20t + 6t^2), \\ E_{3N} &= t \left\{ 4\omega^3 - (7 - 6t)\omega^2 + \frac{1}{6}(29 - 42t + 14t^2)\omega - \frac{1}{6}(18 - 29t + 14t^2 - 2t^3) \right\}, \\ E_{4N} &= t^2 \left\{ -5\omega^4 + (12 - 10t)\omega^3 - \frac{1}{12}(145 - 216t + 75t^2)\omega^2 \right. \\ &\quad \left. + \frac{1}{60}(388 - 725t + 420t^2 - 75t^3)\omega \right. \\ &\quad \left. - \frac{1}{360}(630 - 1164t + 725t^2 - 180t^3 + 15t^4) \right\}, \\ E_{5N} &= t^3 \left\{ 6\omega^5 - \frac{1}{3}(55 - 45t)\omega^4 + \frac{1}{3}(73 - 110t + 39t^2)\omega^3 \right. \\ &\quad \left. - \frac{1}{60}(1094 - 2190t + 1375t^2 - 270t^3)\omega^2 \right. \\ &\quad \left. + \frac{1}{180}(1479 - 3282t + 2555t^2 - 825t^3 + 93t^4)\omega \right. \\ &\quad \left. - \frac{1}{360}(720 - 1479t + 1094t^2 - 365t^3 + 55t^4 - 3t^5) \right\}, \end{aligned}$$

when $N \geq 6$.

We state our next result in this chain of manipulations as follows.

LEMMA 5. *There are functions $G_{pN}(t)$ such that*

$$(1 + E)^{1/2} - 1 = \chi(t) \sum_{p=1}^N G_{pN}(t)(n+1)^{-p}.$$

The function G_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree $2p - 2$ in t when $1 \leq p \leq N - 1$. The function G_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree $2N - 2$ in t whose coefficients are independent of n . In particular,

$$\begin{aligned} G_{1N} &= \frac{1}{2}E_{1N}, \quad G_{2N} = \frac{1}{2}E_{2N} - \frac{1}{8}\chi E_{1N}^2, \\ G_{3N} &= \frac{1}{2}E_{3N} - \frac{1}{4}\chi E_{1N}E_{2N} + \frac{1}{16}\chi^2 E_{1N}^3, \\ G_{4N} &= \frac{1}{2}E_{4N} - \frac{1}{8}\chi(E_{2N}^2 + 2E_{1N}E_{3N}) + \frac{3}{16}\chi^2 E_{1N}^2 E_{2N} - \frac{5}{128}\chi^3 E_{1N}^4, \\ G_{5N} &= \frac{1}{2}E_{5N} - \frac{1}{4}\chi(E_{1N}E_{4N} + E_{2N}E_{3N}) \\ &\quad + \frac{3}{16}\chi^2 E_{1N}(E_{2N}^2 + E_{1N}E_{3N}) - \frac{5}{32}\chi^3 E_{1N}^3 E_{2N} + \frac{7}{256}\chi^4 E_{1N}^5, \end{aligned}$$

when $N \geq 6$.

This follows by a now familiar reasoning from (6) and the identity,

$$(1 + E)^{1/2} - 1 = \sum_{k=1}^N \left\{ \frac{\Gamma(k - \frac{1}{2})(-1)^{k-1} \gamma_{kN}}{2\pi^{1/2} k!} \right\} E^k,$$

in which $\gamma_{kN} = 1$ when $1 \leq k \leq N - 1$ and $\gamma_{NN} = (1 + \theta_4 E)^{-N+1/2}$, that is valid for some θ_4 such that $0 < \theta_4 < 1$. We need to observe that, as a consequence of Lemma 4, $1 + \theta_4 E \geq 0.3492$, $\gamma_{NN} \leq (2.8367)^{N-1/2}$.

3. Proof of the theorem. We observe first that

$$\begin{aligned} \nu_n &= \frac{4}{\pi} \int_0^\delta (1-x^2)^{-1} dx - F + G, \\ (7) \quad \nu_n &= (2/\pi) \log\{(1+\delta)/(1-\delta)\} - F + G, \end{aligned}$$

in which

$$\begin{aligned} F &= \frac{4}{\pi} \int_0^\delta (1-x^2)^{-1} \left[1 - \{1 - h_n^2(x)\}^{1/2} \right] dx \\ &= \frac{4}{\pi} \int_0^\delta (1-x^2)^{-1} h_n^2(x) \left[1 + \{1 - h_n^2(x)\}^{1/2} \right]^{-1} dx, \\ G &= \frac{4}{\pi} \int_\delta^1 (1-x^2)^{-1} \{1 - h_n^2(x)\}^{1/2} dx. \end{aligned}$$

We now show that F is asymptotically negligible.

LEMMA 6. $F = O(a^3 e^{-2a}) = o\{(n+1)^{-N}\}$ for large n and any N .

Here and throughout this section the constants implicit in the symbols O and o may depend on N , but are independent of n . Because $0 \leq h_n(x) \leq (n+1)x^n$,

$$0 < F < \frac{4(n+1)^2}{\pi(1-\delta^2)} \int_0^\delta x^{2n} dx = \frac{4(n+1)^2 \delta^{2n+1}}{\pi(1-\delta^2)(2n+1)}.$$

We observe when $n \geq 8$ that

$$1 - \delta^2 = a^{-1}(2 - a^{-1}) \geq 5/3a, \quad 2(n+1)/(2n+1) \leq 18/17,$$

and

$$\begin{aligned} \delta^{2n+1} &= \exp\{(2n+1) \log(1 - a^{-1})\} < \exp\{-(2n+1)a^{-1}\} \\ &= \exp(-2a + a^{-1}) \leq e^{1/3} e^{-2a}. \end{aligned}$$

The assertion of the lemma is now apparent.

It we make the substitution $x = 1 - t/(n+1)$ in G and use Lemma 2, we see that

$$\begin{aligned} G &= \frac{2}{\pi} \int_0^a [\{2(n+1) - t\}^{-1} + t^{-1}] dt + A_0 - H + I + J + K, \\ (8) \quad G &= (2/\pi) \log\{a/(1+\delta)\} + A_0 - H + I + J + K, \end{aligned}$$

in which A_0 is the constant defined in (3) and

$$\begin{aligned} H &= \frac{2}{\pi} \int_0^a \{2(n+1) - t\}^{-1} \{1 - (1 - t^2 \operatorname{csch}^2 t)^{1/2}\} dt, \\ I &= \frac{2}{\pi} \int_0^a (1 - t^2 \operatorname{csch}^2 t)^{1/2} t^{-1} \{(1 + E)^{1/2} - 1\} dt, \\ J &= \frac{2}{\pi} \int_0^a \{2(n+1) - t\}^{-1} (1 - t^2 \operatorname{csch}^2 t)^{1/2} \{(1 + E)^{1/2} - 1\} dt, \\ K &= \frac{2}{\pi} \int_a^\infty \{1 - (1 - t^2 \operatorname{csch}^2 t)^{1/2}\} t^{-1} dt. \end{aligned}$$

We now show that K is asymptotically negligible.

LEMMA 7. $K = O(ae^{-2a}) = o\{(n+1)^{-N}\}$ for large n and any N .
It is obvious that $0 \leq 1 - (1-s)^{1/2} \leq s$ when $0 \leq s \leq 1$. Therefore,

$$0 < K < \frac{2}{\pi} \int_a^\infty t \operatorname{csch}^2 t \, dt = O(ae^{-2a}).$$

An asymptotic expansion for H can be inferred from the following lemma.

LEMMA 8. If H_p, L and M are defined so that

$$\begin{aligned} H_p &= (2^{1-p} \pi^{-1}) \int_0^\infty \{1 - (1 - t^2 \operatorname{csch}^2 t)^{1/2}\} t^{p-1} \, dt, \\ L &= \sum_{p=1}^{N-1} 2^{1-p} \pi^{-1} (n+1)^{-p} \int_a^\infty \{1 - (1 - t^2 \operatorname{csch}^2 t)^{1/2}\} t^{p-1} \, dt, \\ (9) \quad H &= \sum_{p=1}^{N-1} H_p (n+1)^{-p} - L + M, \end{aligned}$$

then $L = O(e^{-2a})$ and $M = O\{(n+1)^{-N}\}$ for large n .

The lemma follows from the definition of H , and Taylor's Theorem in the form

$$(10) \quad (1-C)^{-1} = \sum_{p=0}^{N-2} C^p + C^{N-1} (1 + \theta_5 C)^{-N}, \quad 0 < \theta_5 < 1,$$

if we observe that $1 - \theta_5 C \geq 1 - C \geq 1 - 1/2a \geq 5/6$ when $n \geq 8$. In fact, $0 < M < (6/5)^N H_N (n+1)^{-N}$. Moreover,

$$\begin{aligned} 0 < L &< \sum_{p=1}^{N-1} 2^{1-p} \pi^{-1} (n+1)^{-p} \int_a^\infty t^{p+1} \operatorname{csch}^2 t \, dt \\ &= \sum_{p=1}^{N-1} (n+1)^{-p} O(a^{p+1} e^{-2a}) = \sum_{p=1}^{N-1} O(a^{1-p} e^{-2a}) = O(e^{-2a}). \end{aligned}$$

An asymptotic expansion for I can be inferred from the following lemma.

LEMMA 9. If I_p, S and T are defined so that

$$\begin{aligned} I_p &= \frac{2}{\pi} \int_0^\infty t^3 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t G_{pN}(t) \, dt, \\ S &= \frac{2}{\pi} \sum_{p=1}^{N-1} (n+1)^{-p} \int_a^\infty t^3 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t G_{pN}(t) \, dt, \\ T &= \frac{2}{\pi} (n+1)^{-N} \int_0^a t^3 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t G_{NN}(t) \, dt, \end{aligned}$$

then I_p is independent of n and N when $1 \leq p \leq N-1$, $S = O(ae^{-2a})$ and $T = O\{(n+1)^{-N}\}$ for large n , and

$$(11) \quad I = \sum_{p=1}^{N-1} I_p (n+1)^{-p} - S + T.$$

The identity (11) is an immediate consequence of Lemma 5. Moreover, $G_{pN}(t) = O(t^{2p-2})$, uniformly in n when $n \geq 8$, for large t , if $1 \leq p \leq N-1$, so that

$$\begin{aligned} S &= \sum_{p=1}^{N-1} (n+1)^{-p} \int_a^\infty O(t^{2p+1} e^{-2t}) dt \\ &= \sum_{p=1}^{N-1} (n+1)^{-p} O(a^{2p+1} e^{-2a}) = O(ae^{-2a}). \end{aligned}$$

Finally, the existence of the polynomial bound for G_{NN} described in Lemma 5 implies that $T = O\{(n+1)^{-N}\}$.

If we combine Lemma 5 and (10) we find that

$$\frac{(1+E)^{1/2} - 1}{(n+1)(1-C)} = \chi(t) \sum_{p=2}^N Q_{pN}(t)(n+1)^{-p},$$

in which Q_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree $2p-4$ in t , when $2 \leq p \leq N-1$, and Q_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree $2N-4$ in t whose coefficients are independent of n . In particular,

$$\begin{aligned} Q_{2N} &= \frac{1}{2}E_{1N}, \quad Q_{3N} = \frac{1}{2}E_{2N} + \frac{1}{4}tE_{1N} - \frac{1}{8}\chi E_{1N}^2, \\ Q_{4N} &= \frac{1}{2}E_{3N} + \frac{1}{4}tE_{2N} + \frac{1}{8}t^2E_{1N} - \frac{1}{4}\chi(E_{2N} + tE_{1N}/4)E_{1N} + \frac{1}{16}\chi^2 E_{1N}^3, \\ Q_{5N} &= \frac{1}{2}E_{4N} + \frac{1}{4}tE_{3N} + \frac{1}{8}t^2E_{2N} + \frac{1}{16}t^3E_{1N} \\ &\quad - \frac{1}{8}\chi(E_{2N}^2 + 2E_{1N}E_{3N} + tE_{1N}E_{2N} + t^2E_{1N}^2/4) \\ &\quad + \frac{1}{16}\chi^2 E_{1N}^2(3E_{2N} + tE_{1N}/2) - \frac{5}{128}\chi^3 E_{1N}^4, \end{aligned}$$

when $N \geq 6$. An argument almost identical to that used in the proof of Lemma 9 now suffices to establish the following result.

LEMMA 10. *If J_p, U and V are defined so that*

$$\begin{aligned} J_p &= \pi^{-1} \int_0^\infty t^4 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t Q_{pN}(t) dt, \\ U &= \pi^{-1} \sum_{p=2}^{N-1} (n+1)^{-p} \int_a^\infty t^4 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t Q_{pN}(t) dt, \\ V &= \pi^{-1} (n+1)^{-N} \int_0^a t^4 (1 - t^2 \operatorname{csch}^2 t)^{-1/2} \operatorname{csch}^2 t Q_{NN}(t) dt, \end{aligned}$$

then J_p is independent of n and N when $2 \leq p \leq N-1$, $U = O(e^{-2a})$ and $V = O\{(n+1)^{-N}\}$ for large n , and

$$(12) \quad J = \sum_{p=2}^{N-1} J_p (n+1)^{-p} - U + V.$$

If we combine (7), (8), (9), (11) and (12), we see that

$$\nu_n = \frac{2}{\pi} \log(n+1) + A_0 + \sum_{p=1}^{N-1} A_p (n+1)^{-p} + O\{(n+1)^{-N}\},$$

in which $A_p = -H_p + I_p + J_p$ (the hitherto undefined quantity $J_1 = 0$). This establishes the existence of the asymptotic expansion (5) of the Theorem, and prescribes the coefficients A_p . There remains only the straightforward, although tedious, verification of the specific results for A_1, A_2, A_3, A_4 and A_5 stated in the Theorem, based on the definitions of H_p, I_p and J_p given in Lemmas 8, 9 and 10. This verification is facilitated by the reduction formula

$$\begin{aligned} \int_0^\infty t^{2q+p+1}(1-t^2\operatorname{csch}^2 t)^{-q-1/2}(t\coth t-1)Z(t)\operatorname{csch}^{2q+2} t \, dt \\ = \int_0^\infty t^{2q+p-1}(1-t^2\operatorname{csch}^2 t)^{-q+1/2}Z_1(t)\operatorname{csch}^{2q} t \, dt, \end{aligned}$$

valid if $1 \leq q$, $Z(t)$ is continuously differentiable on $(0, \infty)$, there is a positive integer m such that $Z(t) = O(t^m)$ and $Z'(t) = O(t^m)$ for large t , $Z(t) = O(t^{2q-p})$ and $Z'(t) = O(t^{2q-p})$ for small t , and

$$Z_1(t) = \{(2q + p - 2qt \coth t)Z(t) + tZ'(t)\}/(2q - 1).$$

Kahaner [10] has tabulated seven decimal place values of ν_n when $n = 1(1)100$. The expansion (5), terminated after the term for which $p = 2$ (or $p = 3$), produces seven decimal place values that either agree with or exceed by at most 10^{-7} those of Kahaner when $n \geq 30$. Inclusion of the term for which $p = 4$ (and $p = 5$) produces the same agreement when $n \geq 8$. The error when $n = 1$ is only 0.0008575. (There is a typographical error in the value ν_{98} reported by Kahaner. The correct value is 3.5510552.)

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