WEIGHTS OF SEMI-INVARIANTS OF THE QUOTIENT DIVISION RING OF AN ENVELOPING ALGEBRA

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ABSTRACT. Let L be a finite dimensional Lie algebra over a field k of characteristic zero, $D(L)$ the quotient division ring of $U(L)$. It is shown that the weights of the semi-invariants of $D(L)$ form a finitely generated, free abelian group $\Lambda_D(L)$. It follows, among other things, that the semicenter of $D(L)$ is isomorphic to the group algebra of $\Lambda_D(L)$ over the center $Z(D(L))$ of $D(L)$.

Introduction. Throughout this paper L will be a nonzero finite-dimensional Lie algebra over a field k of characteristic zero. Let $U(L)$ be the universal enveloping algebra of L with center $Z(U(L))$, $D(L)$ the division ring of quotients of $U(L)$ with center $Z(D(L))$. For each $\lambda \in L^*$ we denote by $D(L)_\lambda$ the set of those $u \in D(L)$ such that $\text{ad } x(u) = \lambda(x)u$ for all $x \in L$. Its elements are called the semi-invariants of $D(L)$ relative to λ .

Clearly $D(L)_\lambda D(L)_\mu \subset D(L)_{\lambda+\mu}$ for all $\lambda, \mu \in L^*$. Put $U(L)_\lambda = U(L) \cap D(L)_\lambda$. The sum of the $D(L)_\lambda$ (resp. $U(L)_\lambda$) is direct and is a factorial subalgebra $Sz(D(L))$ (resp. $Sz(U(L))$) which is called the semicenter of $D(L)$ (resp. $U(L)$) [10, 13]. Let $\Lambda_D(L)$ (resp. $\Lambda(L)$) be the set of all weights $\lambda \in L^*$ such that $D(L)_\lambda \neq 0$ (resp. $U(L)_\lambda \neq 0$). Since any semi-invariant of $D(L)$ can be written as a quotient of two semi-invariants of $U(L)$, $\Lambda_D(L)$ is the additive subgroup of L^* generated by the semigroup $\Lambda(L)$ [18, 6, p. 329]. Using the fact that each subfield of $D(L)$ containing k is finitely generated over k we are able to prove that $\Lambda_D(L)$ is a finitely generated free abelian group. This result has some interesting consequences. For instance, the semicenter $Sz(D(L))$ is isomorphic to the group algebra of $\Lambda_D(L)$ over $Z(D(L))$. In particular $Sz(D(L))$ is Noetherian. The same need not be true however for $Sz(U(L))$ (not even if L is nilpotent). Also, it is shown that there exists a nonzero semi-invariant e in $U(L)$ such that the localization $U(L)_e$ of $U(L)$ satisfies the Formanek condition, i.e. any nonzero ideal of $U(L)_e$ contains a nonzero central element of $U(L)_e$. Moreover, the center $Z(U(L)_e)$ is factorial and its quotient field coincides with $Z(D(L))$. Finally, let $\lambda_1, \dots, \lambda_r$ be a $\mathbf{Z}$ -basis of $\Lambda_D(L)$. If L is Frobenius then $\lambda_1, \dots, \lambda_r$ are linearly independent over k [6, Theorems 3.2 and 4.3]. The same result also holds if L is an algebraic Lie algebra.

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1. The main theorem and some consequences. We thank R. Rentschler for bringing the following result to our attention.

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THEOREM 1.1 [19, THEOREM 3]. *Let D be a division algebra over a field k , F a commutative subfield of D containing k . Then F is a finitely generated field extension of k if and only if $D \otimes_k F$ is left and right Noetherian.*

COROLLARY 1.2. (a) *Let F be a commutative subfield of $D(L)$ containing k . Then F is a finitely generated field extension of k .*

(b) *k is algebraically closed in $D(L)$; that is, if $a \in D(L)$ is algebraic over k then $a \in k$.*

PROOF. The proof is analogous with that of Theorem 4 of [19].

(a) $U(L) \otimes_k F$ is isomorphic to $U(L \otimes_k F)$ and hence is left and right Noetherian. The same holds for $D(L) \otimes_k F$ since it is the localization of $U(L) \otimes_k F$ at the Ore set $\{u \otimes 1 \mid u \in U(L), u \neq 0\}$. Finite generation of F follows from Theorem 1.1.

(b) Let $a \in D(L)$ and put $F = k(a)$. As $D(L) \otimes_k F$ is a localization of $U(L \otimes_k F)$ (up to isomorphism), it is a domain. Since $F \otimes_k F$ embeds in $D(L) \otimes_k F$ it also is a domain. Now, if a is algebraic over k , then $a \in k$ by [5, p. 248 ex. 5].

We can now prove the main result of this note.

THEOREM 1.3. $\Lambda_D(L)$ is a finitely generated free abelian group.

PROOF. Since $\Lambda_D(L)$ is a torsion free abelian group, it suffices to show that it is finitely generated. Let F be the common quotient field of $Sz(U(L))$ and $Sz(D(L))$ [6, Corollary 1.11]. By (a) of the foregoing corollary, F is a finite field extension of k . Hence there exist nonzero semi-invariants $u_i \in U(L)_{\mu_i}$, such that $F = k(u_1, \dots, u_s)$ (we may even assume that the u_i are irreducible). Let G be the subgroup of $\Lambda_D(L)$ generated by $\mu_1, \dots, \mu_s$. We claim that $G = \Lambda_D(L)$. Take any $\lambda \in \Lambda_D(L)$ and $w \in D(L)_\lambda$ such that $w \neq 0$.

We have to show that $\lambda \in G$. We can find nonzero elements $u, v \in k[u_1, \dots, u_s]$ such that $w = uv^{-1}$. Hence $u = \sum_m a_m u_1^{m_1} \cdots u_s^{m_s}$ where $a_m \in k$ and $m = (m_1, \dots, m_s) \in \mathbb{N}^s$. Clearly $a_m u_1^{m_1} \cdots u_s^{m_s} \in U(L)_\mu$ where $\mu = m_1 \mu_1 + \cdots + m_s \mu_s \in G$. After regrouping the terms according to their weights, we obtain $u = \sum x_\alpha$, $\alpha \in G$ and $x_\alpha \in U(L)_\alpha$. Similarly, we have $v = \sum y_\beta$, $\beta \in G$ and $y_\beta \in U(L)_\beta$, not all zero. Next, we observe that

$$\sum w y_\beta = w v = u = \sum x_\alpha; \quad \alpha, \beta \in G,$$

where $w y_\beta \in D(L)_{\lambda+\beta}$, $x_\alpha \in D(L)_\alpha$. From this and the fact that the sum of the $D(L)_\sigma$ is direct, it follows that for any $\beta \in G$ $w y_\beta = x_\alpha$ for a suitable $\alpha \in G$. Now choose $\beta \in G$ such that $y_\beta \neq 0$. Then $\lambda + \beta = \alpha$ and thus $\lambda = \alpha - \beta \in G$.

We are now in a position to completely describe the structure of $Sz(D(L))$.

THEOREM 1.4. *Let $\lambda_1, \dots, \lambda_r$ be a $\mathbb{Z}$ -basis for $\Lambda_D(L)$. Choose nonzero semi-invariants $w_i \in D(L)_{\lambda_i}$. Then the following hold*

1. (a) *For any $\lambda \in \Lambda_D(L)$ we have $D(L)_\lambda = Z(D(L))w_1^{m_1} \cdots w_r^{m_r}$, where the integers m_i are uniquely determined by $\lambda = m_1 \lambda_1 + \cdots + m_r \lambda_r$.*

(b) $w_1, \dots, w_r$ are algebraically independent over $Z(D(L))$.

2. $Sz(D(L))$ is isomorphic to the group algebra $Z(D(L))[\Lambda_D(L)]$.

3. $Sz(D(L)) = Z(D(L))[w_1, \dots, w_r, w_1^{-1}, \dots, w_r^{-1}]$.

In particular, $Sz(D(L))$ is Noetherian and its quotient field is $Z(D(L))(w_1, \dots, w_r)$, a pure transcendental extension of $Z(D(L))$.

PROOF. For each $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$ we put $w^m = w_1^{m_1} \cdots w_r^{m_r}$.

1. (a) If $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r$ then $w^m \in D(L)_\lambda$ and so $Z(D(L))w^m \subset D(L)_\lambda$. On the other hand, let $v \in D(L)_\lambda$. Then $v(w^m)^{-1} \in D(L)_0 = Z(D(L))$ and thus $v \in Z(D(L))w^m$.

(b) (a) and the fact that $Sz(D(L)) = \bigoplus D(L)_\lambda$ show that the w^m , $m \in \mathbf{Z}^r$, form a basis of $Sz(D(L))$ over $Z(D(L))$. In particular they are linearly independent over $Z(D(L))$.

2. Using (1) it is now easy to verify that the map $\phi: Z(D(L))[\Lambda_D(L)] \rightarrow Sz(D(L))$ sending aX_λ into aw^m , where $a \in Z(D(L))$ and $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r$, is an isomorphism of $Z(D(L))$ -algebras.

3. Is a direct consequence of (1).

REMARK 1.5. (a) Let L^∞ be the intersection of the ideals L^n , the lower central series of L . We know that $[L, L^\infty] = L^\infty$ and that L^∞ is the smallest ideal M of L such that L/M is nilpotent. In the next section it is shown that the w_i may always be chosen in $Z(D(L^\infty)) \cap D(L)_\lambda$.

(b) (3) of the previous theorem offers a new proof for the factoriality of $Sz(D(L))$ since it is a localization of $Z(D(L))[w_1, \dots, w_r]$, which is factorial.

We thank Paul Wauters for allowing us to include the following result.

THEOREM 1.6 (P. WAUTERS). *If the semicenter $Sz(U(L))$ is Noetherian, then $\Lambda(L)$ is finitely generated.*

PROOF. For the sake of notation, we put $R = Sz(U(L))$ and if A is an ideal of the semigroup $\Lambda(L)$ then we write $R_A = \bigoplus U(L)_\lambda$, $\lambda \in A$ (see e.g. [9] for generalities on semigroups). Clearly, R_A is an ideal of R . Moreover, if $A_1 \subset \cdots \subset A_n \subset \cdots$ is a strictly ascending chain of ideals of $\Lambda(L)$, then $R_{A_1} \subset \cdots \subset R_{A_n} \subset \cdots$ is a strictly ascending chain of ideals of R . Since R is Noetherian, $\Lambda(L)$ satisfies ACC on ideals. Put $T = k[\Lambda(L)]$, the monoid ring of $\Lambda(L)$ over k , which we view as a $\Lambda_D(L)$ -graded ring. Clearly, every graded ideal of T is of the form $k[A]$ where A is an ideal of $\Lambda(L)$. Hence T satisfies ACC on graded ideals. Since $\Lambda_D(L)$ is a finitely generated abelian group, it follows that T is a Noetherian ring [17, p. 612]. Therefore $\Lambda(L)$ is finitely generated [9, Theorem 20.7].

COROLLARY 1.7. *$Sz(U(L))$ and $Z(U(L))$ need not be Noetherian.*

PROOF. By [8, pp. 320–321] there exists a complex solvable algebraic Lie algebra $\mathfrak{b}$ of dimension 47 for which $\Lambda(\mathfrak{b})$ is not finitely generated. Hence, $Sz(U(\mathfrak{b}))$ is not Noetherian. Moreover, let $\mathfrak{h}$ be the 45 dimensional nilpotent ideal of $\mathfrak{b}$ as defined in [8, p. 320]. Then it is easy to check that $\mathfrak{h}$ coincides with $\mathfrak{b}_\Lambda$, the intersection of all $\ker \lambda$, $\lambda \in \Lambda(\mathfrak{b})$. By [6, Theorem 1.19] it follows that $Z(U(\mathfrak{h})) = Z(U(\mathfrak{b}_\Lambda)) = Sz(U(\mathfrak{b}))$, which is not Noetherian. Since $\Lambda(\mathfrak{h}) = 0$, this example also demonstrates that the converse of Theorem 1.6 does not hold.

2. Properties of the localization $U(L)_e$.

LEMMA 2.1. *Let u be a nonzero semi-invariant of $U(L)$ with weight λ . Then $Z(U(L^\infty))$ contains a nonzero semi-invariant u' of $U(L)$ with the same weight.*

PROOF. We can find ideals H_i of L with $\dim H_i = i$, $[L, H_i] \subset H_{i-1}$ and such that $L^\infty = H_d \subset \cdots \subset H_{i-1} \subset H_i \subset \cdots \subset H_n = L$. We now distinguish two cases.

(1) $u \in U(L^\infty)$. Clearly λ vanishes on $[L, L^\infty] = L^\infty$. Hence $[y, u] = 0$ for all $y \in L^\infty$, which implies that $u \in Z(U(L^\infty))$.

(2) $u \in U(H_i) \setminus U(H_{i-1})$ for some $i > d$. Choose $x \in H_i \setminus H_{i-1}$. Then $U(H_i) = U(H_{i-1})[x]$ and each element of $U(H_i)$ can be written uniquely as a polynomial in x with coefficients in $U(H_{i-1})$ (say on the left) [7, 4.4.4]. In particular $u = a_m x^m + \cdots + a_1 x + a_0$ for some $a_j \in U(H_{i-1})$, $a_m \neq 0$. Take any $y \in L$. Then $\text{ad } y(x) \in H_{i-1}$ and $\lambda(y)u = \text{ad } y(u) = (\text{ad } y(a_m))x^m + \text{terms of lower degree in } x$.

It follows that $\text{ad } y(a_m) = \lambda(y)a_m$ for all $y \in L$, i.e. $a_m \in U(L)_\lambda \cap U(H_{i-1})$. Finally, after repeating the same reasoning a number of times, we find a nonzero semi-invariant u' of $U(L)$ with weight λ , contained in $U(L^\infty)$. Then $u' \in Z(U(L^\infty))$ as in Case 1.

Because of Theorem 1.3, we are able to use an approach similar to that of [2, 6.7]. Let $\lambda_1, \dots, \lambda_r$ be a $\mathbf{Z}$ -basis of the group $\Lambda_D(L)$. Since $\Lambda_D(L)$ is generated by $\Lambda(L)$, there are distinct weights $\mu_1, \dots, \mu_t \in \Lambda(L)$ such that

$$\lambda_i = m_{i1}\mu_1 + \cdots + m_{it}\mu_t$$

for some integers m_{ij} . By the foregoing lemma, there exist nonzero semi-invariants e_i of $U(L)$ with weight μ_i , contained in $Z(U(L^\infty))$. [By replacing the μ_i , we can take the e_i to be irreducible as well.] In particular, $e = e_1 \cdots e_t \in Z(U(L^\infty))$ is a nonzero semi-invariant of $U(L)$ with weight $\mu = \mu_1 + \cdots + \mu_t$. We denote by $U(L)_e$ the localization (which exists) of $U(L)$ with respect to $\{1, e, e^2, \dots\}$. Next we put

$$w_i = e_1^{m_{i1}} \cdots e_t^{m_{it}}.$$

Clearly, $w_i \in Z(D(L^\infty)) \cap D(L)_{\lambda_i}$. Also, $w_i e^p \in U(L)$ for some nonnegative integer p , i.e. $w_i \in U(L)_e$. Similarly, $w_i^{-1} \in U(L)_e$.

THEOREM 2.2. *Put $A = U(L)_e$, $A_\lambda = A \cap D(L)_\lambda$ and $Sz(A) = \bigoplus A_\lambda$, $\lambda \in L^*$. Then the following hold:*

1. *For any $\lambda \in \Lambda_D(L)$ we have $A_\lambda = Z(A)w_1^{m_1} \cdots w_r^{m_r}$, where the integers m_i are uniquely determined by $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r$.*
2. *The semicenter $Sz(A)$ is isomorphic to the group ring $Z(A)[\Lambda_D(L)]$. Furthermore, $Sz(A) = Z(A)[w_1, \dots, w_r, w_1^{-1}, \dots, w_r^{-1}]$.*
3. *$Sz(A) = Sz(U(L))_e$.*
4. *Both $Sz(A)$ and $Z(A)$ are factorial.*
5. *$Z(A) = \{ue^{-m} \mid u \in U(L)_{m\mu}, m \in \mathbf{N}\}$.*

PROOF. (1) and (2) are proved in the same fashion as in Theorem 1.4.

3. Any $v \in A_\lambda$ is of the form $v = ue^{-m}$ where $u \in U(L)$ and m is a non-negative integer. Clearly, $u = ve^m \in U(L) \cap D(L)_{\lambda+m\mu} = U(L)_{\lambda+m\mu}$ and thus $v \in (Sz(U(L)))_e$. This implies that $Sz(A) \subset (Sz(U(L)))_e$. On the other hand let $x \in (Sz(U(L)))_e$. Then $x = (\sum_i x_i)e^{-p}$ where x_i is a semi-invariant of $U(L)$. Hence $x = \sum_i (x_i e^{-p}) \in Sz(A)$.

4. $Sz(A)$ is factorial, being a localization of the factorial ring $Sz(U(L))$. Using (2), the group ring $Z(A)[\Lambda_D(L)]$ is factorial and hence so is $Z(A)$ by a result of Gilmer and Parker [1, Theorem 2.6].

5. From the proof of (3) we deduce that

$$A_\lambda = \{ue^{-m} \mid u \in U(L)_{\lambda+m\mu}, m \in \mathbf{N}\}.$$

On the other hand, $Z(A) = A_0$. This proves (5). We note that $Z(U(L)) \subset Z(A) \subset Z(D(L))$ and $Sz(U(L)) \subset Sz(A) \subset Sz(D(L))$. Next let E be the multiplicative set

of nonzero semi-invariants of $U(L)$. An important result by Moeglin asserts that each nonzero ideal of $U(L)$ contains an element of E [11, 14], i.e. the localization $U(L)_E$ is simple. Also, $U(L)_E$ and $D(L)$ have the same center and semicenter, since any nonzero semi-invariant of $D(L)$ is a quotient of two elements of E .

COROLLARY 2.3. 1. Put $S = Z(A) \setminus \{0\}$. The localization A_S coincides with $U(L)_E$.

2. Any nonzero ideal I of A contains an element of S .

3. $Z(D(L))$ is the quotient field of $Z(A)$.

PROOF. 1. Take $x \in U(L)_E$. Then $x = uv^{-1}$ with $u \in U(L)$, $v \in U(L)_\lambda$, $v \neq 0$. By (1) of the previous theorem, $v = cw$ for some nonzero $c \in Z(A)$ and $w \in A$ such that $w^{-1} \in A$. Clearly, $x = uv^{-1} = (uw^{-1})c^{-1} \in A_S$. This shows that $U(L)_E \subset A_S$. On the other hand, any $y \in A_S$ is of the form $y = as^{-1}$ where $a \in A \subset U(L)_E$ and $s \in S$.

By (5) of the previous theorem, $s = ve^{-m}$ where $v \in E$ and m is a nonnegative integer. Therefore, $s^{-1} = e^m v^{-1} \in U(L)_E$ and thus $y = as^{-1} \in U(L)_E$.

2. Let I be a nonzero ideal of A . Then $I_S = \{us^{-1} \mid u \in I, s \in S\}$ is a nonzero ideal of the simple algebra A_S . Hence, $1 \in I_S$, i.e. $1 = us^{-1}$ for some $u \in I$, $s \in S$. So, $s = u \in I$.

3. Clearly, the quotient field of $Z(A)$, i.e. $Z(A)_S$, coincides with the center of $A_S = U(L)_E$, which is $Z(D(L))$.

3. k -linear independence of weights. First we recall the following well-known result [3, §8, Proposition 1].

LEMMA 3.1. Let V be a vector space over k ; F a subfield of k and W a subspace of V over F . If W has a basis over F which is also a basis of V over k , then V is generated by W over k and every subset of W free over F is also free over k .

We thank Theo Moons for pointing out the following

PROPOSITION 3.2. Suppose L has a decomposition of the form $L = S \oplus N \oplus A$, where S is a semisimple Lie subalgebra, N the nil radical of L and A an Abelian Lie subalgebra such that $[A, S] = 0$ and with respect to some basis $x_1, \dots, x_q$ of N each $\text{ad}_N a$, $a \in A$, has a diagonal matrix, say $\text{diag}(\phi_1(a), \dots, \phi_q(a))$, $\phi_i \in A^*$. (Such a decomposition exists for instance if L is almost algebraic and k is algebraically closed [6, proof of Theorem 1.19.3]). Now, let $\lambda \in \Lambda(L)$ be a weight. Then λ vanishes on $S \oplus N$ and its restriction $\lambda|_A = \sum n_i \phi_i$ for some nonnegative integers n_i .

PROOF. The first part is obvious since $S = [S, S]$ and each $\text{ad } x$, $x \in N$, acts locally nilpotently on $U(L)$. Next, put $B = S \oplus A$, which is a Lie subalgebra of L . Choose a nonzero semi-invariant $u \in U(L)$ with weight λ . By the Poincaré-Birkhoff-Witt theorem, u has a unique expression: $u = \sum u_m x_1^{m_1} \cdots x_q^{m_q}$, $u_m \in U(B)$, $m = (m_1, \dots, m_q)$.

Take any $a \in A$ and $n = (n_1, \dots, n_q)$ such that $u_n \neq 0$. From $\text{ad } a(u) = \lambda(a)u$ and the fact that $\text{ad } a(u_m) = 0$ we deduce $\lambda(a) = \sum n_i \phi_i(a)$.

We call L algebraic if L is isomorphic to an algebraic Lie algebra of linear transformations. See e.g. [4, §14].

THEOREM 3.3. *Let L be algebraic. If $\lambda_1, \dots, \lambda_r \in \Lambda_D(L)$ are linearly independent over $\mathbf{Z}$, then they are also linearly independent over k .*

PROOF. *Case 1.* k is algebraically closed.

Since L is algebraic it is a fortiori almost algebraic [20, §1.3] and hence allows a decomposition $L = S \oplus N \oplus A$ as in the previous proposition. Note that $R = N \oplus A$ is the radical of L and N is the nil radical of R . Let W denote the $\mathbf{Q}$ -subspace of A^* generated by the weights $\phi_1, \dots, \phi_q$ of $\text{ad } A$ on N . Since L is algebraic, so is its radical R [4, p. 309]. Moreover, R is completely solvable, as k is algebraically closed. By [12, Theorem 3.3 and Corollary 3.4] there is a basis $a_1, \dots, a_t$ for A such that $\phi_i(a_j) \in \mathbf{Q}$ for all i, j and $\dim_{\mathbf{Q}} W = \dim_k A^* = t$. We may assume that $\phi_1, \dots, \phi_t$, $t \leq q$, form a basis for W over $\mathbf{Q}$. Then the rows of the $t \times t$ matrix $P = (\phi_i(a_j))$ with rational entries are linearly independent over $\mathbf{Q}$ and so P is nonsingular. This implies that the rows of P are also linearly independent over k . Therefore $\phi_1, \dots, \phi_t$ are linearly independent over k and thus form a basis for A^* over k . By Lemma 3.1, every subset of W free over $\mathbf{Q}$ is also free over k . On the other hand, Proposition 3.2 asserts that

$$\{\lambda|_A | \lambda \in \Lambda(L)\} \subset \left\{ \sum_{i=1}^q n_i \phi_i | n_i \in \mathbf{N} \right\}$$

which implies that $\{\lambda|_A | \lambda \in \Lambda_D(L)\} \subset W$. Since $\lambda_1, \dots, \lambda_r \in \Lambda_D(L)$ vanish on $N \oplus S$ and are linearly independent over $\mathbf{Z}$ (and hence over $\mathbf{Q}$) the same is true of $\lambda_1|_A, \dots, \lambda_r|_A \in W$. By the preceding observation $\lambda_1|_A, \dots, \lambda_r|_A$ are linearly independent over k and the same a fortiori holds for $\lambda_1, \dots, \lambda_r$.

Case 2. Let k be arbitrary and denote by k' its algebraic closure. Then $\lambda_1, \dots, \lambda_r$ can be extended to weights $\lambda'_1, \dots, \lambda'_r$ of $L' = L \otimes k'$, which is an algebraic Lie algebra over k' .

$\lambda'_1, \dots, \lambda'_r$ are linearly independent over $\mathbf{Z}$, hence also over k' (Case 1). Consequently, $\lambda_1, \dots, \lambda_r$ are linearly independent over k .

REMARK 3.4. The following example demonstrates that the previous theorem no longer holds in the almost algebraic case. Let L be the Lie algebra over $\mathbf{C}$ with basis x, u_1, u_2 and nonvanishing brackets: $[x, u_1] = (1+i)u_1$, $[x, u_2] = (1-i)u_2$. L is a solvable, almost algebraic Lie algebra, u_1, u_2 are semi-invariance with weights λ_1, λ_2 . Clearly λ_1, λ_2 are linearly independent over $\mathbf{Z}$, but not over $\mathbf{C}$.

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