RADIAL SYMMETRY OF THE FIRST EIGENFUNCTION FOR THE p -LAPLACIAN IN THE BALL

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ABSTRACT. We prove the radial symmetry of the eigenfunction corresponding to the first eigenvalue of the equation: $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, when Ω is a ball in R^n and $1 < p < \infty$.

Introduction. Let Ω be a bounded domain in R^n , $n \geq 2$. Consider the problem

$$(1_\lambda) \quad \begin{aligned} \Delta_p u + \lambda|u|^{p-2}u &= 0 \quad \text{in } \Omega, \\ u &\in W_0^{1,p}(\Omega), \quad u \not\equiv 0, \quad \lambda \in R, \quad \text{and } 1 < p < \infty, \end{aligned}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian of u .

We say that u is a solution of (1_λ) if there exists a λ , and (1_λ) holds in the sense of distributions, i.e.

$$(1'_\lambda) \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \psi = \lambda \int_\Omega |u|^{p-2}u\psi \quad \forall \psi \in W_0^{1,p}(\Omega).$$

Consider the following minimization problem

$$(2) \quad \inf I(v), \quad v \in W_0^{1,p}(\Omega), \quad J(v) = 1,$$

where $I(v) = (1/p) \int_\Omega |\nabla v|^p$ and $J(v) = (1/p) \int_\Omega |v|^{p-1}v$.

It can be shown [1, 2] that there exists a smallest $\lambda_1 > 0$ and $u_{\lambda_1} > 0$ that solves (1_{λ_1}) . In that case λ_1 is the infimum in (2). Furthermore u is a solution (2) iff u solves (1_{λ_1}) and satisfies $J(u) = 1$. In the case Ω is a ball, the spherically decreasing rearrangements of positive solutions are also solutions. Furthermore if $\phi(r)$ is a radial solution then all other radial solutions are scalar multiples of $\phi(r)$ [1]. The radial solution $\phi(r)$ can be chosen to be positive.

In [1] the question is raised as to whether or not a solution, corresponding to λ_1 in a ball, must be radially symmetric. In the following we prove that if u solves (1_{λ_1}) in a ball, then u is radially symmetric (see Theorem 1). Our method is based on an idea of Pólya and Szegő [4, p. 89].

Radial symmetry. Let Ω be the ball of radius R centered at the origin. Let $u \in W_0^{1,p}(\Omega)$ solve (1_{λ_1}) . By a remark at the end of Theorem 4 in [5, p. 264], for $1 < p \leq n$, $\|u\|_{\infty, \Omega} \leq C\|u\|_{p, \Omega}$. If $p > n$, by Theorem 7.17 in [3, p. 163] $u \in C^\beta(\bar{\Omega})$ with $\beta = 1 - n/p$.

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Thus $u \in L^\infty(\Omega)$ and by the results in [7], $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$. Let $\phi(r) > 0$ be a radial solution of (1_{λ_1}) and let f be such that

$$(3) \quad u = f\phi.$$

By Theorem 3 in [1], $\phi \in C^1(\overline{\Omega})$ and hence $f \in C^1(\Omega)$.

Let $\text{vol}(S)$ denote the volume of a region S in R^n and $\lambda_1(S)$ denote the infimum of (2) in S .

We now prove a few preliminary results that lead to the main theorem. We begin with

PROPOSITION 1. *Let Ω_1 and Ω_2 be balls of radius R_1 and R_2 centered at the origin with $R_1 < R_2$. Then $\lambda_1(\Omega_2) < \lambda_1(\Omega_1)$.*

PROOF. Let ϕ_1 be radial and minimize (2) in Ω_1 . We take $\phi_1 > 0$. Then $I(\phi_1) = \lambda_1(\Omega_1)$ and $J(\phi_1) = 1$.

Set $\hat{\phi}(r) = c^{n/p}\phi_1(cr)$ where $c = R_1/R_2$. Then $\hat{\phi}(R_2) = 0$, $J(\hat{\phi}_2) = 1$, and $I(\hat{\phi}_2) = c^p\lambda_1(\Omega_1)$. But $\lambda_1(\Omega_2)$ is the infimum of (2) in Ω_2 and hence $\lambda_1(\Omega_2) \leq c^p\lambda_1(\Omega_1) < \lambda_1(\Omega_1)$.

LEMMA 1. *If $u \in W_0^{1,p}(\Omega)$ solves (1_{λ_1}) , then either $u > 0$ or $u < 0$ in Ω .*

PROOF. First we show that u does not change sign. Suppose it does. Let $u^+ = \max\{u, 0\}$ and $\Omega^+ = \{x \in \Omega : u > 0\}$. As u^+ is continuous, Ω^+ is an open subset of Ω with $0 < \text{vol}(\Omega^+) < \text{vol}(\Omega)$. Furthermore, $u^+ \in W_0^{1,p}(\Omega^+)$. Scale u^+ so that $J(u^+) = 1$ in Ω^+ . Putting u^+ as ψ in $(1'_{\lambda_1})$ gives $I(u^+) = \lambda_1(\Omega) \geq \lambda_1(\Omega^+)$. Let Ω_s be the ball centered at the origin with $\text{vol}(\Omega_s) = \text{vol}(\Omega^+)$. By Theorem 2 in [1] (also see [6]), $\lambda_1(\Omega^+) \geq \lambda_1(\Omega_s)$. Proposition 1 then implies $\lambda_1(\Omega) \geq \lambda_1(\Omega^+) \geq \lambda_1(\Omega_s) > \lambda_1(\Omega)$, a contradiction.

To prove that u never vanishes in Ω , we use the strong maximum principle as stated in Proposition 3.2.2 in [8].

Note. From now on we will take $u > 0$. It then follows from (3) that $f > 0$ in Ω .

For notational simplicity we write

$$\|u\|_p = \|u\|_{p,\Omega}, \quad 1 \leq p \leq \infty.$$

LEMMA 2. *If $u \in W_0^{1,p}(\Omega)$ solves (1_{λ_1}) and $\phi(r) > 0$ is a radial solution of (1_{λ_1}) , and if f is such that $u = f\phi$, then $f \in L^\infty(\Omega)$.*

PROOF. Now $f = u/\phi$ where $\phi(r)$ is never zero. Let Ω_1 be the ball of radius R_1 centered at the origin with $0 < R_1 < R$. Then

$$\sup_{\Omega_1} f \leq \|u\|_\infty / \inf_{\Omega_1} \phi \leq M < +\infty.$$

Consider the annulus $A = \Omega \setminus \overline{\Omega}_1$. Define $w(r) = k(R-r)$ for $k > 0$ where $r = |x|$. Then

$$\Delta_p w = -(n-1)k^{p-1}/r.$$

Choose k so large such that

$$(4) \quad -(n-1)k^{p-1}/R < -\lambda_1\|u\|_\infty^{p-1} \quad \text{and} \quad k(R-R_1) > \|u\|_\infty.$$

Then $\Delta_p w \leq \Delta_p u$ with $w \geq u$ on ∂A . By the weak comparison principle, $w \geq u$ in A . Thus $u(x) \leq k(R-|x|)$ with k as chosen in (4). By Theorem 3 in [1] and

Proposition 3.2.1 in [8], there is a $c > 0$ such that $\phi'(r) \leq -c < 0$ in $R_1 \leq r \leq R$. Hence, $\phi(r) \geq c(R - r)$ in A . This implies $f = u/\phi \leq k/c < +\infty$ in A . Thus $f \in L^\infty(\Omega)$.

LEMMA 3. We have $f^p \phi \in W_0^{1,p}(\Omega)$.

PROOF. It is obvious that $f^p \phi \in L^p(\Omega)$. Now

$$\nabla(f^p \phi) = f^p \nabla \phi + p \phi f^{p-1} \nabla f.$$

Also

$$\nabla f = \frac{\nabla u}{\phi} - \frac{u}{\phi^2} \nabla \phi = \frac{\nabla u}{\phi} - f \frac{\nabla \phi}{\phi}.$$

Hence

$$(5) \quad \phi \nabla f = \nabla u - f \nabla \phi.$$

The above yields

$$|\nabla(f^p \phi)| \leq f^p |\nabla \phi| + p f^{p-1} \{|\nabla u| + f |\nabla \phi|\}.$$

As $f, \phi, \nabla \phi \in L^\infty(\Omega)$ and $u, \phi \in W_0^{1,p}(\Omega)$, it follows that $f^p \phi \in W^{1,p}(\Omega)$. We now show $f^p \phi$ is the strong limit in $W^{1,p}(\Omega)$ of a sequence of functions belonging to $C_0^1(\Omega)$.

For $n = 2, 3, 4, \dots$, let $R_n = (n-1)R/n$ and $\bar{R}_n = R_n + R/2n$, and Ω_n be the ball of radius R_n centered at the origin. For each n , let $0 \leq h_n(r) \leq 1$ be a radial function in $C_0^1(\Omega)$ such that

$$h_n(r) = \begin{cases} 1 & \text{in } 0 \leq r \leq R_n, \\ 0 & \text{in } \bar{R}_n \leq r \leq R, \end{cases}$$

and $|\nabla h_n| \leq 8n/R$. Then the functions $g_n = (f^p \phi) h_n$ belong to $C_0^1(\Omega)$. We will show that the functions g_n converge to $f^p \phi$ in $W^{1,p}(\Omega)$.

By Theorem 3 in [1], $\phi \in C^1(\bar{\Omega})$ and ϕ vanishes on $\partial\Omega$. Thus there exists a $c > 0$ such that $\phi'(r) \geq -c$ in $R/2 \leq r \leq R$. Integrating, we obtain that $\phi(r) \leq c(R - r)$ in $R/2 \leq r \leq R$. In particular, it follows that $\phi(r) \leq c(R - R_n)$ in $R_n \leq r \leq \bar{R}_n$. This implies that

$$(6) \quad \phi(r) \leq cR/n \quad \text{in } R_n \leq r \leq \bar{R}_n.$$

We first show that $g_n \rightarrow f^p \phi$ in $L^p(\Omega)$ as $n \rightarrow \infty$. We have

$$\begin{aligned} \int_{\Omega} |g_n - f^p \phi|^p &= \int_{\Omega} (f^p \phi)^p |1 - h_n|^p \leq \|f^p \phi\|_{\infty}^p \int_{\Omega \setminus \Omega_n} 1 \\ &\leq \|f^p \phi\|_{\infty}^p \text{vol}(\Omega \setminus \Omega_n). \end{aligned}$$

The result is obtained by noting that $\text{vol}(\Omega \setminus \Omega_n) \rightarrow 0$ as $n \rightarrow \infty$. Next we show that $\nabla g_n \rightarrow \nabla(f^p \phi)$ in $L^p(\Omega)$. By Minkowski's inequality

$$\begin{aligned} \left(\int_{\Omega} |\nabla g_n - \nabla(f^p \phi)|^p \right)^{1/p} &\leq \left(\int_{\Omega} |\nabla(f^p \phi)|^p |1 - h_n|^p \right)^{1/p} + \left(\int_{\Omega} (f^p \phi)^p |\nabla h_n|^p \right)^{1/p} \\ &\leq \left(\int_{\Omega \setminus \Omega_n} |\nabla(f^p \phi)|^p \right)^{1/p} + \|f^p \phi\|_{\infty} \left(\int_{\Omega \setminus \Omega_n} \phi^p |\nabla h_n|^p \right)^{1/p}. \end{aligned}$$

The first term on the right side of the inequality can be made small by the absolute continuity of the integral. We estimate the second term as follows. Using (6), we obtain that

$$\phi^p |\nabla h_n|^p \leq (cR/n)^p (8n/R)^p \leq M < +\infty \quad \forall n = 2, 3, 4, \dots$$

Hence

$$\int_{\Omega \setminus \Omega_n} \phi^p |\nabla h_n|^p \leq M \operatorname{vol}(\Omega \setminus \Omega_n).$$

Since $\operatorname{vol}(\Omega \setminus \Omega_n) \rightarrow 0$ as $n \rightarrow \infty$, we obtain that $\nabla g_n \rightarrow \nabla(f^p \phi)$ in $L^p(\Omega)$. This completes the proof.

LEMMA 4. *The following integrals exist:*

- (i) $I_1 = \int_{\Omega} f^p |\nabla \phi|^p < \infty$,
- (ii) $I_2 = \int_{\Omega} |f^{p-2} \phi (\nabla \phi \cdot \nabla f)| |\nabla \phi|^{p-2} < \infty$,
- (iii) $I_3 = \int_{\Omega} \phi^p |\nabla f|^p < \infty$.

PROOF. We note using (5) that

$$|\phi (\nabla \phi \cdot \nabla f)| \leq |\nabla \phi| \{ |\nabla u| + f |\nabla \phi| \}$$

and

$$\phi^2 |\nabla f|^p \leq (|\nabla u| + f |\nabla \phi|)^p.$$

Thus I_1, I_2, I_3 exist, as $f, \phi, \nabla \phi \in L^\infty(\Omega)$ and $u, \phi \in W_0^{1,p}(\Omega)$.

PROPOSITION 2. *Let A and B be vectors in R^n , $n \geq 2$. If $1 < p < \infty$, then*

$$(7) \quad |A + B|^p \geq |A|^p + p|A|^{p-2} A \cdot B + K|B|^p$$

where $|V|$ denotes the length of a vector V , and

$$K = \begin{cases} \min \left(1, \frac{|A + B|^p - |A|^p - p|A|^{p-2} A \cdot B}{|B|^p} \right); & B \neq 0, \\ 0, & B = 0. \end{cases}$$

Furthermore, $0 \leq K \leq 1$ and $K = 0$ iff $B = 0$.

Before proving Proposition 2, we prove the following Lemma.

LEMMA 5. *If $t \geq 0$, α , and s are any real numbers, and if $1 < p < \infty$, then*

$$(8) \quad \{(1 + \alpha)^2 t^2 + s^2\}^{p/2} \geq (1 + \alpha p) t^p.$$

Equality holds in (8) iff $s = 0$ and either $\alpha = 0$ or $t = 0$.

PROOF. Observe that for all s ,

$$(9) \quad ((1 + \alpha)^2 t^2 + s^2)^{p/2} \geq |1 + \alpha|^p t^p.$$

To prove (8), it is then sufficient to prove that for all α

$$(10) \quad |1 + \alpha|^p \geq 1 + \alpha p.$$

Case 1. If $1 + \alpha \geq 0$, then $|1 + \alpha|^p = (1 + \alpha)^p$. Define

$$F(\alpha) = (1 + \alpha)^p - 1 - \alpha p \quad \forall \alpha \geq -1.$$

Then

$$F'(\alpha) = p(1 + \alpha)^{p-1} - p.$$

It is easy to see that $F'(\alpha) < 0$ when $-1 \leq \alpha < 0$, and $F'(\alpha) > 0$ when $\alpha > 0$. Noting that $F(0) = 0$, we obtain (10). It is clear that, in the case $1 + \alpha \geq 0$, equality in (10) holds, iff $\alpha = 0$.

Case 2. If $1 + \alpha < 0$, then $1 + \alpha p < 0$ so (10) follows trivially, and strict inequality holds.

To prove the 'only if' part of the last assertion of the lemma, we note that for equality in (8), we must have equality in (9). This implies that $s = 0$. If $t \neq 0$, then equality must hold in (10), which implies that $\alpha = 0$.

PROOF OF PROPOSITION 2. If $A = 0$, the inequality in (7) follows easily. Let us assume that $A \neq 0$. Then we can write $B = \alpha A + D$, for some scalar α and some vector D orthogonal to A . Such a decomposition is unique. Then

$$|A + B|^p = ((1 + \alpha)^2 |A|^2 + |D|^2)^{p/2}$$

and

$$|A|^p + p|A|^{p-2} A \cdot B = (1 + \alpha p)|A|^p.$$

From Lemma 5

$$|A + B|^p \geq |A|^p + p|A|^{p-2} A \cdot B.$$

where equality holds iff $B = 0$. Thus, K in (7) is strictly positive whenever $B \neq 0$. This finishes the proof.

We now prove the main theorem of this paper.

THEOREM 1. *Let Ω be a ball in R^n and $u \in W_0^{1,p}(\Omega)$ solve (1_{λ_1}) . Then u is radially symmetric.*

PROOF. By the note at the end of Lemma 1, $u > 0$, and hence (2) yields

$$\int_{\Omega} |\nabla u|^p = \lambda_1 \int_{\Omega} u^p.$$

Using (3), we get

$$\int_{\Omega} |\nabla(f\phi)|^p = \lambda_1 \int_{\Omega} (f\phi)^p.$$

By Lemma 3, $f^p\phi$ is a legitimate test function in $(1'_{\lambda_1})$. Using $\psi = f^p\phi$, and replacing u by ϕ , $(1'_{\lambda_1})$ gives

$$\int_{\Omega} |\nabla\phi|^{p-2} \nabla\phi \cdot \nabla(f^p\phi) = \lambda_1 \int_{\Omega} (f\phi)^p.$$

Hence,

$$(11) \quad \int_{\Omega} |\nabla(f\phi)|^p = \int_{\Omega} |\nabla\phi|^{p-2} \nabla\phi \cdot \nabla(f^p\phi).$$

Now,

$$\int_{\Omega} |\nabla(f\phi)|^p = \int_{\Omega} |f\nabla\phi + \phi\nabla f|^p.$$

Setting $A = f\nabla\phi$ and $B = \phi\nabla f$, and using (7), we have

$$\int_{\Omega} |\nabla(f\phi)|^p \geq \int_{\Omega} f^p |\nabla\phi|^p + p f^{p-1} \phi |\nabla\phi|^{p-2} (\nabla\phi \cdot \nabla f) + K \phi^p |\nabla f|^p,$$

where K is as defined in Proposition 2. Combining the first two terms on the right side of the inequality we obtain

$$(12) \quad \int_{\Omega} |\nabla(f\phi)|^p \geq \int_{\Omega} |\nabla\phi|^{p-2} \nabla\phi \cdot \nabla(f^p\phi) + K\phi^p |\nabla f|^p.$$

Comparing (11) and (12), we get

$$\int_{\Omega} K\phi^p |\nabla f|^p \leq 0.$$

As $\phi > 0$ in Ω , we obtain $K|\nabla f|^p = 0$. From Proposition 2, K and ∇f vanish together. Hence f is a constant and u is radially symmetric.

Note added in proof. The author has been able to extend this method to prove that λ_i is simple for C^2 domains. The details are worked out in his Ph.D. thesis supervised by A. Weitsman. He has also learned that S. Sakaguchi and A. Anane have obtained similar results.

REFERENCES

1. F. De Thelin, *Sur l'espace propre associé à la première valeur propre du pseudo-laplacien*, Comptes Rendus **303** (1986), 355–358.
2. —, *Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linéaire*, Comptes Rendus **299** (1984), 911–914.
3. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin and New York, 1983.
4. G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Studies, no. 27, Princeton Univ. Press, Princeton, N.J., 1951.
5. J. Serrin, *Local behaviour of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.
6. E. Sperner, Jr., *Symmetrisierung für Funktionen mehrerer reeller Variablen*, Manuscripta Math. **11** (1974), 159–170.
7. P. Tolksdorf, *Regularity for more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.
8. —, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), 773–817.

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