ESTIMATES OF POSITIVE LINEAR OPERATORS ON L^p

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ABSTRACT. Let $0 < \alpha < 1$ and T a positive linear operator on L^p , $1 < p < +\infty$, such that $\|(1 - \alpha)I + \alpha T\|_p \leq 1$ (I = identity). For such operators, (which do not necessarily satisfy

(i) $\|T\|_p \leq 1$ (contraction),

(ii) $\sup_{n \geq 1} \|(I + T + \cdots + T^{n-1})/n\|_p \leq 1$ (Cesàro mean bounded by one))

[1] we show, using M. A. Akcoglu's estimate, that

$$\left\| \sup_{\substack{n \geq 1 \\ n \in \mathbf{N}}} \frac{f + Tf + \cdots + T^{n-1}f}{n} \right\|_p \leq \gamma(\alpha) \|f\|_p \quad \text{for any } f \in L^p.$$

We also obtain the pointwise ergodic theorem in L^p .

Introduction. Let (Ω, A, μ) be a σ -finite measure space, $L^p(\Omega, A, \mu)$ (or simply L^p) the usual Banach spaces constructed on (Ω, A, μ) and T a positive linear operator on L^p ($1 < p < +\infty$).

We set $M_n(T) = (I + T + \cdots + T^{n-1})/n$, the Cesàro averages of an operator T and consider for $0 < \alpha < 1$ the class C_α of those operators on L^p which satisfy $\|(1 - \alpha)I + \alpha T\|_p \leq 1$. The purpose of this paper is to obtain an L^p estimate for the supremum of the Cesàro averages of a positive linear operator which belongs to at least one class C_α .

The class C_1 is the set of positive contractions on L^p . For such operators Akcoglu showed [1] that

$$\left\| \sup_{\substack{n \in \mathbf{N} \\ n \geq 1}} \frac{f + Tf + \cdots + T^{n-1}f}{n} \right\|_p \leq \frac{p}{p-1} \|f\|_p.$$

Contractions belong of course to each class C_α and operators which verify $\sup_n \|(I + T + \cdots + T^{n-1})/n\|_p \leq 1$ belong to $C_{1/2}$. But the converse is not true (see [4, Proposition 1]). For positive operators satisfying

$$\sup_{\substack{n \in \mathbf{N} \\ n \geq 1}} \left\| \frac{1 + T + \cdots + T^{n-1}}{n} \right\|_p \leq 1$$

Emilion obtained a dominated estimate [4]. His proof rests on the estimate on C_1 and Hille-Yosida's theorem for resolvents. Our method, which proves more, is independent of the theory of resolvents.

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Ergodic properties of operators in the class C_α . We list first some properties of classes C_α .

PROPOSITION 1. (1) If $0 < \alpha < \beta < 1$ then $C_\beta \subsetneq C_\alpha$.

(2) C_α is convex.

(3) For $p = 2$, $\alpha = 1/2$ there exists $T \in C_{1/2}$ such that $\sup_{n \geq 1} \|M_n(T)\|_2 > 1$.

PROOF. (1) The inclusion $C_\beta \subset C_\alpha$ is easy. To prove that C_α is strictly bigger than C_β we can take $\Omega = \{1, 2\}$, $\mu\{1\} = \mu\{2\} = 1$ and $T = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$.

(2) is easy.

(3) Take $\Omega = \{1, 2, 3\}$, $\mu\{1\} = \mu\{2\} = \mu\{3\} = 1$ and

$$T = \begin{pmatrix} 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 \end{pmatrix}.$$

LEMMA 2. Let $0 < \alpha < 1$. Then there exists $\gamma(\alpha) > 0$ such that

$$\frac{\alpha^n}{n!} \sum_{j=1}^{[n/\alpha]+1} e^{-\alpha j} j^n > \gamma(\alpha) \quad \text{for any } n \in \mathbf{N}^*.$$

PROOF. It suffices to prove that

$$\frac{\alpha^n}{n!} \int_0^{[n/\alpha]+1} e^{-\alpha x} \cdot x^n dx > \gamma(\alpha) \quad \forall n \in \mathbf{N}^*$$

or

$$\frac{1}{n!} \int_0^{\alpha[n/\alpha]+\alpha} e^{-x} \cdot x^n dx > \gamma(\alpha) \cdot \alpha.$$

This inequality follows from [3, p. 704]: $\exists c > 0$ such that $(1/n!) \int_0^n e^{-x} x^n dx > c$, $\forall n \in \mathbf{N}^*$. So we can take $\gamma(\alpha) = c/\alpha$.

THEOREM 3. Let $0 < \alpha < 1$. There exists $\tilde{\gamma}(\alpha) > 0$ such that for any $T \in C_\alpha$ and $f \in L^p$ ($1 < p < +\infty$) we have

$$\left\| \sup_{n \geq 1} M_n(T)f \right\|_p \leq \tilde{\gamma}(\alpha) \cdot \frac{p}{p-1} \|f\|_p.$$

PROOF. We will prove first that for any $T \in C_\alpha$ there exists $S \in C_1$ such that

$$(**) \quad I + T + T^2 + \cdots + T^n \leq (\gamma(\alpha))^{-1} (I + S + \cdots + S^{[n/\alpha]+1}).$$

Take $S = e^{-1} \exp((1-\alpha)I + \alpha T)$ where

$$\exp U = I + \frac{U}{1!} + \frac{U^2}{2!} + \cdots + \frac{U^2}{n!} + \cdots.$$

We have $S = e^{-\alpha} \exp \alpha T$, $S \in C_1$ and $S^n = e^{-n\alpha} \exp(n\alpha T)$. To prove (**) it suffices to show that

$$\forall k \leq n \quad \sum_{j=1}^{[n/\alpha]+1} e^{-\alpha j} \frac{j^k \alpha^k}{k!} \geq \gamma(\alpha).$$

This will be true if

$$\forall n \in \mathbf{N}^*, \quad \sum_{j=1}^{[n/\alpha]+1} \frac{e^{-\alpha j} \cdot j^n \cdot \alpha^n}{n!} \geq \gamma(\alpha).$$

But this relation is true by Lemma 2. So we have

$$\begin{aligned} & (\gamma(\alpha))^{-1} \cdot \frac{[n/\alpha] + 2}{n + 1} \left[\frac{I + S + \dots + S^{[n/\alpha]+1}}{[n/\alpha] + 2} \right] \\ & \geq \frac{I + T + T^2 + \dots + T^n}{n + 1}, \quad \forall n \in \mathbf{N} \end{aligned}$$

and

$$\sup_{n \geq 1} M_n(T)f \leq \tilde{\gamma}(\alpha) \cdot \sup_{n \geq 1} M_n(S)f \quad \forall f \in L^{p+}.$$

The dominated estimate follows from [1].

To get the pointwise convergence we can use the following result, mentioned in [2] even in the L^1 case.

PROPOSITION 4. *Let T be a positive operator on L^p , $1 < p < +\infty$ such that $\sup_{n \geq 1} \|M_n(T)\|_p < +\infty$ and f a positive function in L^p . If $\sup_n M_n(T)f \in L^p$ then $M_n(T)f$ converges a.e. to a fixed function in L^p .*

PROOF. We just need to prove that $g^* = g_*$ where $g^* = \lim_n M_n(T)g_1$, $g_* = \lim_n M_n(T)g_2$, $g_1 = \limsup M_n(T)f$ and $g_2 = \liminf M_n(T)f$. We use a decomposition of Ω in two disjoint parts Y and Z , [5].

(i) for any Y_0^* in $L^q(1/p + 1/q = 1)$, such that $T^*(Y_0^*) = Y_0^*$ then $\text{supp } Y_0^* \subset Y$ and for any f in L^p $\mathbf{1}_Y \cdot M_n(T)f$ converges a.e.

(ii) for any h in L^p such that $\text{supp } h \subset Z$ then $\text{supp } Th \subset Z$ and $\|M_n(T)h\|_p \rightarrow 0$.

For v_0^* in L^q , such that $T^*(v_0^*) = v_0^*$ and $\text{supp } v_0^* = Y$ we have using (i) for any h in L^{p+} ,

$$\int v_0^* \cdot T(\mathbf{1}_Z \cdot h) d\mu = 0 \quad \text{so} \quad \mathbf{1}_Y Th = \mathbf{1}_Y \cdot T(\mathbf{1}_Y \cdot h).$$

Then for any positive integer n

$$\begin{aligned} 0 & \leq \mathbf{1}_Y \cdot T^n(g^* - g_*) = \mathbf{1}_Y \cdot T^n(\mathbf{1}_Y(g^* - g_*)) = \mathbf{1}_Y \cdot (g^* - g_*) \\ & = \mathbf{1}_Y \lim_n M_n(T)(g_1 - g_2) = \mathbf{1}_Y \lim_n M_n(T)(\mathbf{1}_Y(g_1 - g_2)) \end{aligned}$$

but

$$\begin{aligned} \mathbf{1}_Y(g_1 - g_2) &= \mathbf{1}_Y \left(\limsup_n M_n(T)f - \liminf_n M_n(T)f \right) \\ &= 0 \quad \text{as on } Y, M_n(T)f \text{ converges a.e.} \end{aligned}$$

Then $\mathbf{1}_Y(g^* - g_*) = 0$ and $T(g^* - g_*) = T(\mathbf{1}_Z(g^* - g_*)) = \mathbf{1}_Z(g^* - g_*)$. We use now (ii) to get

$$\lim_n \|M_n(T)(\mathbf{1}_Z(g^* - g_*))\|_p = \|\mathbf{1}_Z(g^* - g_*)\|_p = 0.$$

So $g^* = g_*$.

THEOREM 5. *Let $0 < \alpha < 1$ then for any $T \in C_\alpha$, $M_n(T)f$ converges a.e. to a fixed function f^* in L^p .*

PROOF. As we get a dominated estimate, for any f in L^{p+} $M_n(T)f$ converges a.e. by Proposition 4. The conclusion follows easily from the decomposition $f = f^+ - f^-$.

REMARK 6. If $p = 1$ then for any $0 < \alpha < 1$ the class C_α is exactly the set of positive contractions in L^1 .

It would be interesting to know what initial condition on a positive operator T implies that $\sup_n \|T^n\|_p < +\infty$.

REFERENCES

1. M. A. Akcoglu, *A pointwise ergodic theorem in L^p spaces*, Canad. J. Math. **27** (1975), 1075–1082.
2. I. Assani, *Sur la théorie ergodique des opérateurs dans les espaces L^p* , Séminaire d'Initiation à l'Analyse (G. Choquet, J. St. Raymond, M. Rogalski) Communication 1, 1984–1985.
3. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
4. R. Emilion, *On the pointwise ergodic theorem in L^p , $1 < p < +\infty$* , Studia Math. **81** (1985), 171–178.
5. R. Sato, *Ergodic theorems for semi-groups in L^p , $1 < p < +\infty$* , Tôhoku Math. J. **26** (1974), 73–76.

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