TOEPLITZ SPECTRAL INCLUSION AND GENERALIZED IN MODULUS PROPERTY

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ABSTRACT. Let $A \subset C_b(X)$ be a function algebra and let μ be a Borel measure on X and $H^2(\mu) = \overline{A}$ —the closure in $L^2(\mu)$. It turns out that the spectral inclusion theorem for Toeplitz operators, defined in the above context, implies the density of finite sums $\sum_i |u_i|^2$, $u_i \in A$, in

- (i) the cone of positive functions in $L^1(\mu)$,
- (ii) the cone of positive functions in $L^p(\mu)$, $p \leq 2$, if $\mu(X) < \infty$,
- (iii) the cone of positive functions in $C(X)$, if X is compact.

Let X be a locally compact Hausdorff space and let $A \subset C_b(X)$ be a function algebra on X , here $C_b(X)$ stands for the algebra of all continuous and bounded functions on X . Suppose we are given a positive Borel measure μ on X such that $A \subset L^2(\mu)$. Denote by $H^2(\mu)$ the closure of A in $L^2(\mu)$. For $\varphi \in L^\infty(\mu)$ the Toeplitz operator T_φ is defined by

$$T_\varphi f = P(\varphi \cdot f), \quad f \in H^2(\mu),$$

where P is the orthogonal projection from $L^2(\mu)$ onto $H^2(\mu)$. In our previous papers [3, 4] we have noticed, for a class of function algebras, so called spectral inclusion property for T_φ ,

$$\sigma(T_\varphi) \supset \{\text{range } \varphi\},$$

where $\sigma(T_\varphi)$ is the spectrum of T_φ . Namely we have assumed that X was compact and the set of all finite sums $\sum_i |u_i|^2$, $u_i \in A$, was dense in the cone of positive and continuous functions on X . We called such A an algebra with generalized approximation in modulus property.

In this note we analyse more closely the relationship between spectral inclusion for Toeplitz operators and generalized approximation in modulus property. Let us describe this more precisely. Suppose that for any $\varphi \in L^\infty_{\mathbf{R}}(\mu)$ -real functions, $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$, then the finite sums $\sum_i |u_i|^2$, $u_i \in A$, are dense in $L^1_+(\mu)$ —the cone of positive functions in $L^1(\mu)$. In the case $\mu(X) < \infty$, $\varphi \in L^q_{\mathbf{R}}(\mu)$, $q \geq 2$ one can also define an unbounded (densely defined) Toeplitz operator T_φ by the analogous formula. Moreover, in this case spectral inclusion $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$ implies density of $\sum_i |u_i|^2$, $u_i \in A$ in $L^p_+(\mu)$, $1 \leq p \leq 2$. If X is compact and $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$ for any finite, Borel measure $\mu \geq 0$, then A has generalized in modulus property. An application of the first above result to harmonic analysis on a class of LCA groups is shown. The connection of the recent notion of the right triples (A, X, μ) introduced by A. B. Aleksandrov [1] and generalized in modulus property of A is also given at the end of this note.

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In what follows $\|\cdot\|_p$ stands for $L^p(\mu)$ norm.

PROPOSITION 1. *If for every $\varphi \in L^\infty_{\mathbf{R}}(\mu)$ $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$, then for any $\varepsilon > 0$ and any $g \in L^1_+(\mu)$ there exists a finite number of functions $g_1, \dots, g_n \in A$ such that $\|\sum_i |g_i|^2 - g\|_1 < \varepsilon$ and $\|\sum_i |g_i|^2\|_1 \leq \|g\|_1$.*

PROOF. Assume there exist $f_0 \in L^1_+(\mu)$ and $\varepsilon_0 > 0$ such that the ball $B(f_0, \varepsilon_0)$ in $L^1(\mu)$ has empty intersection with the set

$$S = \left\{ \sum_i |h_i|^2 \text{ a finite sums, } h_i \in A, \left\| \sum_i |h_i|^2 \right\|_1 \leq \|f_0\|_1 \right\}.$$

By the separation theorem for convex sets there exists $g_0 \in L^\infty(\mu)$ such that

$$(*) \quad \text{Re} \int g_0 h \, d\mu \geq \alpha > \text{Re} \int g_0 f \, d\mu$$

for all $f \in B(f_0, \varepsilon_0)$ and all $h \in S$. Take $k = \|f_0\|^{1/2} h_1$, where $h_1 \in A$, $\|h_1\|_2 = 1$. Note that $|k|^2 \in S$. Hence by $(*)$ we have

$$\|f_0\| \text{Re} \int g_0 |h_1|^2 \, d\mu \geq \alpha,$$

so

$$(**) \quad \text{Re}(T_{g_0} h_1, h_1) \geq \alpha \|f_0\|^{-1}, \quad \text{for any } h_1 \in A \text{ and } \|h_1\|_2 = 1.$$

Denote by $W(T_\varphi)$ the numerical range of T_φ . We have

$$\{\text{range Re } g_0\} \subset \sigma(T_{\text{Re } g_0}) \subset \overline{W(T_{\text{Re } g_0})} \subseteq \left[\frac{\alpha}{\|f_0\|}, \|T_{\text{Re } g_0}\| \right]$$

the last inclusion following from $(**)$. Thus

$$\text{Re } g_0 \geq \alpha \|f_0\|^{-1}$$

and so we have obtained a contradiction

$$\alpha > \text{Re} \int g_0 f_0 \, d\mu \geq \alpha \|f_0\|^{-1} \int f_0 \, d\mu = \alpha$$

and this completes the proof.

The above result has an application in harmonic analysis. Let G be a locally compact abelian group with the dual $\hat{G}$ and let Σ be a fixed subsemigroup of $\hat{G}$. Denote by μ and $\hat{\mu}$ the normalized Haar measures on G and $\hat{G}$, respectively. Following [2] assume that Σ generates $\hat{G}$ and $\hat{\mu}(\Sigma) > 0$. Denote by $F: L^2(G) \rightarrow L^2(\hat{G})$ the Fourier transform and let

$$H^2(\Sigma) = \{f \in L^2(G), \text{supp } Ff \subseteq \Sigma\}.$$

Now define

$$A = \{g, g = F^{-1}h, \text{supp } h \text{ is a compact set in } \Sigma\}.$$

It is obvious that $A \subset L^2(\mu)$ had $\overline{A} = H^2(\Sigma)$. For a bounded measurable function φ on G the Wiener-Hopf operator defined on $H^2(\Sigma)$ by

$$W_\varphi f = P(\varphi \cdot f),$$

here $P: L^2(G) \rightarrow H^2(\Sigma)$ is the orthogonal projection. By Theorem 1 of [2] we know that $\sigma(W_\varphi) \supset \{\text{range } \varphi\}$. Applying our Proposition 1 we have the following corollary.

COROLLARY. For G, Σ, μ, A as above the set of finite sums $\sum_i |b_i|^2$, $b_i \in A$ is dense in $L_+^1(G)$.

As we have noticed above, in the case $\mu(X) < \infty$ one can prove more. Namely we have

PROPOSITION 2. If $\mu(X) < \infty$ and for any $\varphi \in L_{\mathbf{R}}^2(\mu)$, $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$, then the set $\sum_i |u_i|^2$, $u_i \in A$ is dense in $L_+^p(\mu)$, for all $1 \leq p \leq 2$.

PROOF. One can repeat the reasoning of Proposition 1. Indeed, since $L^p(\mu) \subset L^1(\mu)$ one can assume that the ball $B(f_0, \varepsilon_0)$ in $L^p(\mu)$ has empty intersection with the set S . By the separation theorem for convex sets there exists $g_0 \in L^q(\mu)$, $p^{-1} + q^{-1} = 1$ such that $(*)$ holds. But $q \geq 2$, so $g \in L^2(\mu)$ and the proof goes exactly as before.

REMARK 1. If we knew that T_φ were densely defined for all $\varphi \in L^s(\mu)$, $1 < s < 2$, then the above proof would work for all $1 \leq p < \infty$.

Suppose now that X is compact. It may happen that the spectral inclusion holds for any finite $\mu \geq 0$. In this case we have

PROPOSITION 3. Let X be a compact Hausdorff space. Suppose that for any finite Borel measure $\mu \geq 0$ and $\varphi \in L_{\mathbf{R}}^\infty(\mu)$ $\sigma(T_\varphi) \supset \{\text{range } \varphi\}$. Then A has generalized approximation in modulus property.

PROOF. By Proposition 1 we know that for any finite Borel measure $\mu \geq 0$ any $\varepsilon > 0$ and any $f \in C_+(X)$ there exists a finite number of functions $u_i \in A$ such that $\|f - \sum_i |u_i|^2\|_1 < \varepsilon$. Hence the cone of finite sums $\sum_i |u_i|^2$ is weakly dense in $C_+(X)$ and so by Mazur's theorem is also dense in sup norm in $C_+(X)$.

We conclude this note with the following

REMARK 2. Let X be a compact space and let μ be a positive Borel measure on X . Suppose we are given a function algebra $A \subset C(X)$. In recent paper A. B. Aleksandrov introduced the following notion: a triple (A, X, μ) is called right if for every positive and continuous function φ there exists a sequence $f_n \in A$ such that $\lim_n |f_n| = \varphi$, μ -almost everywhere, and $|f_n| \leq \varphi$, [1].

He proved that (X, A, μ) is right iff there exists $\tau > 0$ such that

$$(+)\quad \sup_{f \in A, |f| < \varphi} \int |f|^2 d\mu \geq \tau \int \varphi^2 d\mu$$

for every positive $\varphi \in C(X)$.

It is clear that this notion depends on the choice of μ . However it may also happen that the above τ does not depend on μ . In this case by the proof of Proposition 9 given in [1] one can find a sequence $f_n \in A$, $|f_n| < \varphi$, $\lim_n |f_n| = \varphi$, ν -a.e., for any Borel measure ν on X . Hence $|f_n| \rightarrow \varphi$ weakly in $C(X)$ and so some convex combinations of the $|f_n|$'s tend to φ uniformly in $C(X)$. In this way we see that A has generalized approximating in modulus property if $(+)$ holds with τ independent on μ .

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