REPRESENTATION OF BOREL ISOMORPHISM BY A PROBABILITY MEASURE

R. M. SHORTT

(Communicated by R. Daniel Mauldin)

ABSTRACT. A theorem of Chuaqui on the representation of equivalence relations by probability measures is applied to Borel isomorphism. Using a result of Cenzer and Mauldin, we show that under $V = L$, the isomorphism types of analytic sets cannot be linearly ordered.

Let X and Y be topological spaces. We indicate their *Borel isomorphism types* by $t(X)$ and $t(Y)$, writing $t(X) \leq t(Y)$ in case there is a Borel-isomorphism of X onto a Borel subset of Y . The usual Cantor-Schroeder-Bernstein construction shows that $\leq$ is a partial order on the class of all types [2, Theorem 2.5]. Given $n = 1, 2, \dots, \omega$ and $t = t(X)$, define $nt = t(nX)$, where nX is the topological sum of n copies of X . Put

$$\text{sec}(t) = \{t(B) : B \text{ Borel subset of } X\}.$$

Then $\text{sec}(t)$, under $\leq$, is a partially ordered set with largest element t .

A topological space is *Polish* if it is separable and completely metrizable. If A and B are Borel subsets of a Polish space P , then $t(A) = t(B)$ if and only if A and B have the same cardinality [5, 8.3.6]. It follows that for $t = t(S)$, S an infinite Borel subset of P , we have $t = \omega t$.

A metric space is *analytic* if it is a continuous image of a Polish space. There is an analytic space U such that $AN = \text{sec}(t(U))$ contains the isomorphism types of all analytic spaces: U is a "universal" analytic set [5, 8.2.16]. The structure of the poset $(AN, \leq)$ has been the object of much study. See [1] for references.

Lusin [7] discovered that there are at least two uncountable analytic isomorphism types, viz. $t(\mathbf{R})$ and $t(U)$. Harrington and Steel [6, 8] showed that there are exactly two if and only if all analytic games are determined, e.g. if there is a measurable cardinal. In particular, this implies that $(AN, \leq)$ is linearly ordered. In [1], Cenzer and Mauldin proved that under $V = L$, $|AN| = 2^\omega$ and there is an uncountable analytic type t with $nt \neq mt$ for $n \neq m$ positive integers. In particular, it was shown that $t \neq \omega t$. Using a result of Chuaqui, we will show how this phenomenon excludes the possibility that $\leq$ linearly orders AN .

Let $(X, \mathcal{B})$ be a measurable space and let $\sim$ be an equivalence relation on $\mathcal{B}$. Then $\sim$ is *refining* if whenever $A \sim B$ are sets in $\mathcal{B}$ such that

$$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset, \quad A_1, A_2 \in \mathcal{B},$$

Received by the editors July 6, 1987 and, in revised form, September 24, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 04A15; Secondary 28A05, 54H05.

Key words and phrases. Analytic set, Borel isomorphism, Borel set.

then there are sets B_1, B_2 in $\mathcal{B}$ such that

$$B = B_1 \cup B_2, \quad B_1 \cap B_2 = \emptyset, \quad A_1 \sim B_1, \quad A_2 \sim B_2.$$

Say that $\sim$ is *countably additive* if whenever $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are each pair-wise disjoint sequences of sets in $\mathcal{B}$ with $A_n \sim B_n$ for $n \geq 1$, then $\bigcup A_n \sim \bigcup B_n$. A set $A \in \mathcal{B}$ is *negligible* for $\sim$ if there is a pairwise disjoint sequence $A_1 A_2 \dots$ in $\mathcal{B}$ such that $A \sim A_n$ for each $n \geq 1$. Define a relation R on $\mathcal{B}$ by allowing $(A, B) \in R$ just in case there is some $C \in \mathcal{B}$ with $C \subseteq B$ and $A \sim C$.

THEOREM (CHUAQUI). *Let $(X, \mathcal{B})$ be a measurable space and let $\sim$ be a countably additive and refining equivalence relation on $\mathcal{B}$ such that*

- (1) *X is not negligible for $\sim$;*
- (2) *the relation R linearly orders the n equivalence classes.*

Then there is a unique probability measure μ on $(X, \mathcal{B})$ such that

- (a) *$\mu(A) = 0$ if and only if $A \in \mathcal{B}$ is negligible for $\sim$;*
- (b) *if $\mu(A) \neq 0$, then $\mu(A) = \mu(B)$ if and only if $A \sim B$.*

INDICATION. This is Theorem 2.11 in [3]. Compare also [4].

Call a metric space X *universally measurable (u.m.)* if for each Borel probability measure μ on X , there is a σ -compact set $S \subseteq X$ with $\mu(S) = 1$. Note that each such S is a Borel set in its completion. Thus, if S is infinite, and $t = t(S)$, then $t = \omega t$.

Each analytic space is u.m. [5, 8.4.1]. The u.m. subsets of an uncountable Polish space P form a σ -field containing the σ -field generated by the analytic subsets of P .

Our main result applies not only to analytic spaces, but to u.m. spaces in general.

PROPOSITION. *Let X be an infinite u.m. space with $t = t(X)$. If $\text{sec}(t)$ is linearly ordered, then $t = \omega t$.*

DEMONSTRATION. Assume rather that $t < \omega t$. Consider the σ -field $\mathcal{B}(X)$ under the equivalence relation $\sim$ defined by

$$A \sim B \Leftrightarrow A, B \text{ Borel isomorphic.}$$

The relation $\sim$ is countably additive and refining. Because $t < \omega t$, we have that X is not negligible. This and the fact that $\text{sec}(t)$ is linearly ordered allows the application of Chuaqui's theorem.

Thus there is a Borel probability measure μ on X such that $\mu(A) = 0$ if and only if A is negligible. Because X is u.m., there exists an infinite σ -compact $S \subseteq X$ with $\mu(S) = 1$. However, $t(S) = \omega t(S) \leq t$, so that each such S is negligible, a contradiction. Q.E.D.

COROLLARY. *Under $V = L$, the poset $(\mathcal{AN}, \leq)$ is not linearly ordered.*

It should be noted that the corollary can be obtained directly from the methods of [1]. In fact, the sets $Cd[n]$ defined on p. 16 of [1] are pairwise incomparable.

REFERENCES

1. D. Cenzer and R. D. Mauldin, *Borel equivalence and isomorphism of coanalytic sets*, *Dissertationes Math.* **228** (1984), 1-32.
2. J. P. R. Christensen, *Topology and Borel structure*, North-Holland and Elsevier, Amsterdam and New York, 1974.

3. R. Chuaqui, *Cardinal algebras and measures invariant under equivalence relations*, Trans. Amer. Math. Soc. **142** (1969), 61–79.
4. R. Chuaqui and J. Malitz, *Preorderings compatible with probability measures*, Trans. Amer. Math. Soc. **279** (1983), 811–824.
5. D. L. Cohn, *Measure theory*, Birkhäuser, Boston, Mass., 1980.
6. L. Harrington, *Analytic determinacy and $0^\#$* , J. Symbolic Logic **43** (1978), 685–693.
7. N. N. Lusin, *Les ensembles analytiques*, 2nd ed., Chelsea, New York, 1972.
8. J. Steel, *Analytic sets and Borel isomorphism*, Fund. Math. **108** (1980), 82–88.

DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457