AMENABILITY AND DERIVATIONS OF THE FOURIER ALGEBRA

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ABSTRACT. It is shown that a locally compact group G is amenable if and only if every derivation of the Fourier algebra $A(G)$ into a Banach $A(G)$ -bimodule is continuous. Also given are necessary and sufficient conditions for $A(G)$ to be weakly amenable.

Introduction. If G is an abelian locally compact group, then it is known that every derivation of $A(G)$, the Fourier algebra of G , into a Banach $A(G)$ -bimodule is continuous [8, p. 410]. We shall prove in the first part of this paper that this automatic continuity property for derivations of $A(G)$ characterizes amenable locally compact groups.

In the second part of this paper we will examine the derivations of $A(G)$ into commutative Banach $A(G)$ -bimodules. We shall focus our attention on the nature of $VN(G)$, the dual of $A(G)$, as a commutative Banach $A(G)$ -bimodule and, in particular, on the $A(G)$ -submodule $UCB(\hat{G})$ of $VN(G)$.

$UCB(\hat{G})$ is the norm closure in $VN(G)$ of the linear span of $A(G) \cdot VN(G)$. The definition of $UCB(\hat{G})$ is due to E. Granirer, who studied its properties in [5]. Recently, Bade, Curtis, and Dales [1] introduced the notion of a weakly amenable Banach algebra $\mathcal{A}$ as one in which every continuous derivation of $\mathcal{A}$ into a commutative Banach $\mathcal{A}$ -bimodule is zero. We will prove that $A(G)$ is weakly amenable if and only if every continuous derivation of $A(G)$ into $UCB(\hat{G})$ is zero. We also present evidence to suggest that $A(G)$ may well be weakly amenable for a large class of locally compact groups. In particular, if G is discrete, then $A(G)$ is weakly amenable.

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Definitions and notation. Let G be a locally compact group with a fixed left Haar measure dx . For each complex-valued function f on G and every $x \in G$ define $L_x f(y) = f(x^{-1}y)$, $f^\vee(x) = f(x^{-1})$, and $\tilde{f}(x) = \overline{f(x^{-1})}$.

G is said to be amenable if there exists $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1_G) = 1$ and $m(L_x f) = m(f)$ for every $x \in G$, $f \in L^\infty(G)$.

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The Fourier algebra of G , denoted by $A(G)$, is the linear subspace of $C_0(G)$ (the continuous complex-valued functions on G which vanish at infinity) consisting of all functions $(f * \tilde{g})^\vee$, where $f, g \in L^2(G)$. $VN(G)$ is the closure in the weak operator topology of the linear span of $\{L_x; x \in G\}$ in $B(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$. $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$ [3, pp. 210 and 218]. With pointwise multiplication and $\|u\|_{A(G)} = \sup\{|\langle T, u \rangle|; T \in VN(G), \|T\| \leq 1\}$, $A(G)$ is a Banach algebra with spectrum $\Delta(A(G)) = G$ [3, p. 222].

If I is an ideal of $A(G)$, let $Z(I) = \{x \in G; u(x) = 0 \text{ for every } u \in I\}$. If $A \subset G$ is closed, then $I(A) = \{u \in A(G); u(x) = 0 \text{ for every } x \in A\}$ is a closed ideal of $A(G)$. A is called a set of spectral synthesis (or simply an S -set) if the only closed ideal I in $A(G)$ with $Z(I) = A$ is $I(A)$. An ideal I is called cofinite if $\text{codim } I = \dim(A(G)/I) < \infty$.

An algebraic bimodule X of the Banach algebra $\mathcal{A}$ is called a Banach $\mathcal{A}$ bimodule if X is a Banach space, $\|u \cdot x\| \leq \|u\| \|x\|$ and $\|x \cdot u\| \leq \|u\| \|x\|$ for every $u \in \mathcal{A}$, $x \in X$. X is commutative if $x \cdot u = u \cdot x$ for every $u \in \mathcal{A}$, $x \in X$.

$VN(G)$ becomes a commutative Banach $A(G)$ -bimodule by means of the formula $\langle u \cdot T, v \rangle = \langle T \cdot u, v \rangle = \langle T, uv \rangle$ for every $T \in VN(G)$, $u, v \in A(G)$. If $T \in VN(G)$, then $\text{supp } T = \{x \in G; u(x) = 0 \text{ for every } u \in A(G) \text{ with } u \cdot T = 0\}$. A map $\Gamma: VN(G) \rightarrow VN(G)$ is said to be invariant if $u \cdot \Gamma(T) = \Gamma(u \cdot T)$ for every $u \in A(G)$, $T \in VN(G)$.

$P(G)$ denotes the continuous positive definite functions on G .

Continuity of derivations on $A(G)$.

DEFINITION 1. Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$ -bimodule. A derivation $D: \mathcal{A} \rightarrow X$ is a linear map which satisfies

$$D(uv) = u \cdot D(v) + D(u) \cdot v \quad \text{for every } u, v \in \mathcal{A}.$$

LEMMA 1. Let G be a nonamenable locally compact group. Then there exists a discontinuous derivation of $A(G)$ into a finite-dimensional commutative Banach $A(G)$ -bimodule.

PROOF. If G is nonamenable, then $I^2(\{e\}) = \{\sum_{i=1}^n u_i v_i; u_i, v_i \in I(\{e\})\}$ is not closed in $A(G)$ [4, Lemma 6.7]. Therefore, either $I^2(\{e\})$ is not cofinite or $I^2(\{e\})$ is cofinite but not closed. In either case, there exists a nonclosed cofinite ideal I of $A(G)$ such that $I^2(\{e\}) \subseteq I \subset I(\{e\})$. Then, as is shown in [2, p. 402], the finite-dimensional space $I(\{e\})/I$ can be made into a commutative Banach $A(G)$ -bimodule and a discontinuous derivation $D: A(G) \rightarrow I(\{e\})/I$ can be constructed. $\square$

LEMMA 2. Let G be an amenable locally compact group. Let I be a closed ideal in $A(G)$ with infinite codimension. Then there exist sequences $\{u_n\}$, $\{v_n\}$ in $A(G)$ such that $u_n v_1 \cdots v_{n-1} \notin I$ but $u_n v_1 \cdots v_n \in I$ for all $n \geq 2$.

PROOF. Let $A = Z(I)$. Since G is amenable and I has infinite codimension, A is infinite [4, Corollary 5.6].

Assume that x is a cluster point of A . Let $x_1 \in A$, $x_1 \neq x$. Let V_1 be a compact neighborhood of x_1 with $x \notin V_1$. Choose $x_2 \in A \setminus V_1$, $x_2 \neq x$. Let V_2 be a compact neighborhood of x_2 such that $x \notin V_2$ and $V_1 \cap V_2 = \emptyset$. Proceeding inductively,

we get $x_n \in A \setminus \bigcup_{i=1}^{n-1} V_i$, $x_n \neq x$ and a compact neighborhood V_n of x_n such that $x \notin V_n$ and $V_n \cap (\bigcup_{i=1}^{n-1} V_i) = \emptyset$.

If A has no cluster points, then there exists a discrete set $\{x_1, x_2, \dots\} \subseteq A$. Let V_1 be a compact neighborhood of x_1 with $\{x_2, x_3, \dots\} \cap V_1 = \emptyset$. With V_{n-1} chosen, let V_n be a compact neighborhood of x_n such that $V_n \cap \{x_{n+1}, x_{n+2}, \dots\} = \emptyset$ and $V_n \cap (\bigcup_{i=1}^{n-1} V_i) = \emptyset$.

In either case, we get a sequence $\{V_n\}$ of compact neighborhoods of points $\{x_n\}$ in A such that $V_n \cap (\bigcup_{i=1}^{n-1} V_i) = \emptyset$ for $n \geq 2$.

For each $n = 1, 2, \dots$, let u_n be a function $A(G)$ with $u_n \geq 0$, $\text{supp } u_n \subseteq V_n$, $u_n(x_n) > 0$ and $\|u_n\|_{A(G)} = 1/2^n$.

For $k = 1, 2, \dots$ define

$$v_k = \sum_{i=k+1}^{\infty} u_i.$$

Since $\sum_{k=1}^{\infty} \|u_k\|_{A(G)} < \infty$, each $v_k \in A(G)$.

If $n \geq 2$, then

$$u_n v_1 \cdots v_{n-1}(x_n) > 0.$$

Hence $u_n v_1 \cdots v_{n-1} \notin I$. However, $\text{supp } u_n \subseteq V_n$ and $v_n(V_n) = 0$. Therefore,

$$u_n v_1 \cdots v_n = 0 \in I. \quad \square$$

THEOREM 1. *Let G be a locally compact group. Then the following are equivalent:*

- (i) G is amenable.
- (ii) Every derivation of $A(G)$ into a finite-dimensional commutative Banach $A(G)$ -bimodule is continuous.
- (iii) Every derivation of $A(G)$ into a Banach $A(G)$ -bimodule is continuous.

PROOF. That (iii) implies (ii) is trivial. That (ii) implies (i) follows immediately from Lemma 1.

Assume that G is amenable. Let I be a closed cofinite in $A(G)$. By [4, Corollary 6.6], $I^2 = I$. This, together with Lemma 2 and [6, Theorem 2], implies that every derivation from $A(G)$ into a Banach $A(G)$ -bimodule is continuous. That is, (i) implies (iii). $\square$

Weak amenability of $A(G)$.

DEFINITION 2. Let φ be a multiplicative linear functional on a Banach algebra $\mathcal{A}$. A point derivation at φ is a linear function $d: \mathcal{A} \rightarrow \mathbb{C}$ such that

$$d(uv) = \varphi(u) d(v) + \varphi(v) d(u) \quad \text{for every } u, v \in \mathcal{A}.$$

PROPOSITION 1. *Let G be a locally compact group. Then $A(G)$ has no continuous nonzero point derivations at any point in the spectrum of $A(G)$.*

PROOF. Let $x \in G = \Delta(A(G))$; cf. [3, p. 229]. Let d be a continuous point derivation at x . Let $v \in A(G)$ be such that $v(x) = 1$ and $\|v\|_{A(G)} = 1$. Then

$$d(v^n) = nd(v) \quad \text{for } n = 1, 2, \dots$$

Since d is bounded, $d(v) = 0$. Let $v_1, v_2 \in I(\{x\})$. Then

$$d(v_1 v_2) = v_1(x) d(v_2) + v_2(x) d(v_1) = 0.$$

$I^2(\{x\})$ is an ideal in $A(G)$ with $Z(I^2(\{x\})) = \{x\}$. As $\{x\}$ is an S -set [3, p. 229], $I^2(\{x\})$ is dense in $I(\{x\})$. Therefore

$$d(u) = 0 \quad \text{for every } u \in I(\{x\}).$$

However, if $u \in A(G)$, then $u = u(x)v + (u - u(x)v)$ and

$$d(u) = u(x)d(v) + d(u - u(x)v) = 0. \quad \square$$

LEMMA 3. *Let G be a locally compact group. Let $x \in G$. Then there exists a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle = \{\lambda L_x; \lambda \in \mathbb{C}\}$.*

PROOF. We follow an idea of P. Renaud [7]. Let $\{V_\alpha\}_{\alpha \in \mathfrak{A}}$ be a neighborhood basis at e . Let $u_\alpha \in P(G) \cap A(G)$ be such that $u_\alpha(e) = 1$ and $\text{supp } u_\alpha \subseteq V_\alpha$. Let $m \in VN(G)^*$ be a weak-* cluster point of $\{L_x u_\alpha\}$. Then, as in [7, Proposition 3 and Theorem 4],

$$m(u \cdot T) = u(x)m(T) \quad \text{for every } u \in A(G), T \in VN(G).$$

Furthermore, $m(L_x) = 1$. Therefore, define

$$P(T) = m(T)L_x.$$

P is an invariant projection of $VN(G)$ onto $\langle L_x \rangle$. $\square$

LEMMA 4. *Let $D: A(G) \rightarrow VN(G)$ be a continuous derivation. Let $X \in G$ and let P_x be a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle$. Define*

$$D_{P_x}(u) = P_x(D(u)) \quad \text{for every } u \in A(G).$$

Then D_{P_x} is a continuous point derivation at x . In particular, $D_{P_x} = 0$.

PROOF. Let $u, v \in A(G)$. Then

$$\begin{aligned} D_{P_x}(uv) &= P_x(D(uv)) = P_x(u \cdot D(v) + v \cdot D(u)) \\ &= u \cdot P_x(D(v)) + v \cdot P_x(D(u)) \\ &= u(x)D_{P_x}(v) + v(x)D_{P_x}(u). \quad \square \end{aligned}$$

Let X be a commutative Banach $A(G)$ -bimodule and let $D: A(G) \rightarrow X$ be a continuous derivation. For each $\varphi \in X^*$ and $u \in A(G)$ define $T_{\varphi, u} \in VN(G)$ by

$$\langle T_{\varphi, u}, v \rangle = \varphi(v \cdot D(u)) \quad \text{for every } v \in A(G).$$

The main idea in the next lemma is due to Bade, Curtis, and Dales [1].

LEMMA 5. *Let X be a commutative Banach $A(G)$ -bimodule and let $D: A(G) \rightarrow X$ be a continuous derivation. For each $\varphi \in X^*$ and $u \in A(G)$, define $\tilde{D}_\varphi(u) = T_{\varphi, u}$. Then $\tilde{D}_\varphi: A(G) \rightarrow VN(G)$ is a continuous derivation. Furthermore, if D is nonzero, then for some $\varphi \in X^*$, $\tilde{D}_\varphi$ is nonzero.*

PROOF. Let $u, v, w \in A(G)$.

$$\begin{aligned} (\tilde{D}_\varphi(uv))(w) &= \varphi(w \cdot D(uv)) = \varphi(w \cdot u \cdot D(v) + w \cdot v \cdot D(u)) \\ &= (\tilde{D}_\varphi(v))(uw) + (\tilde{D}_\varphi(u))(vw) \\ &= [u \cdot \tilde{D}_\varphi(v)](w) + [v \cdot \tilde{D}_\varphi(u)](w). \end{aligned}$$

Hence $\tilde{D}_\varphi$ is a derivation. Since $\varphi \in X^*$ and D is continuous, $\tilde{D}_\varphi$ is also continuous.

As $A(G)^2$ is dense in $A(G)$ [3, p. 223], if D is nonzero, there exist $u, v \in A(G)$ such that $D(uv) \neq 0$. However,

$$uv = \frac{(u+v)^2 - u^2 - v^2}{2},$$

so we may assume that $D(u^2) \neq 0$ for some $u \in A(G)$. Let $\varphi \in X^*$ be such that $\varphi(D(u^2)) \neq 0$. Then

$$[\tilde{D}\varphi(u)](u) = \varphi(u \cdot D(u)) = \frac{1}{2}\varphi(D(u^2)) \neq 0. \quad \square$$

LEMMA 6. *Let $D: A(G) \rightarrow VN(G)$ be a continuous derivation. Then $D(u) \in UCB(\hat{G})$ for every $u \in A(G)$.*

PROOF. Let $u, v \in A(G)$. Then $D(uv) = uD(v) + vD(u) \in UCB(\hat{G})$. Since $A^2(G)$ is dense in $A(G)$ and $UCB(\hat{G})$ is closed, $D(u) \in UCB(\hat{G})$ for every $u \in A(G)$. $\square$

THEOREM 2. *Let G be a locally compact group. Then the following are equivalent:*

- (i) $A(G)$ is weakly amenable.
- (ii) Every continuous derivation from $A(G)$ into $UCB(\hat{G})$ is zero.

PROOF. That (i) implies (ii) is trivial.

Conversely, it follows from Lemma 5 and Lemma 6 that if $A(G)$ is not weakly amenable, then there exists a nonzero continuous derivation of $A(G)$ into $UCB(\hat{G})$. $\square$

Theorem 2 is a stronger version of [1, Theorem 1.5].

PROPOSITION 2. *Let $D: A(G) \rightarrow UCB(\hat{G})$ be a continuous derivation and let $u \in A(G)$. Then $\text{supp } D(u)$ contains no isolated points.*

PROOF. Assume that V is an open neighborhood of $x \in \text{supp } D(u)$ with $V \cap \text{supp } D(u) = \{x\}$. Let $v \in A(G)$ be such that $v(x) = 1$ and $\text{supp } v \subseteq V$. Define $D_0: A(G) \rightarrow UCB(\hat{G})$ by $D_0(w) = v \cdot D(w)$ for every $w \in A(G)$. Then D_0 is a continuous derivation. As $v(x) = 1$, $x \in \text{supp } D_0(u)$ [3, p. 225]. But $\text{supp } v \cdot D(u) \subseteq \text{supp } v \cap \text{supp } D(u) = \{x\}$. Therefore $\text{supp } D_0(u) = \{x\}$ and $D_0(u) = \lambda L_x$ for some $\lambda \neq 0$ [3, p. 229].

By lemma 3, there exists a continuous invariant projection P_x of $VN(G)$ onto $\langle L_x \rangle$. By lemma 4, $P_x \circ D_0$ is a nonzero continuous point derivation of $A(G)$ at x , contradicting Proposition 1. $\square$

THEOREM 3. *Let G be a discrete group. Then $A(G)$ is weakly amenable.*

PROOF. This follows immediately from Theorem 2 and Proposition 2 if we note that $\text{supp } T = \emptyset$ if and only if $T = 0$ [3, p. 224]. $\square$

THEOREM 4. *Let G be a discrete group. Then the following are equivalent:*

- (i) G is amenable.
- (ii) Every derivation from $A(G)$ into a finite-dimensional commutative Banach $A(G)$ -bimodule is zero.

PROOF. If G is amenable, then every derivation is continuous (Theorem 1). Therefore (ii) holds by Theorem 3. If G is nonamenable, then by Lemma 1, there

exists a discontinuous derivation of $A(G)$ into a commutative finite-dimensional Banach $A(G)$ -bimodule. $\square$

It seems that $A(G)$ is weakly amenable for every locally compact group G (or at least for all amenable locally compact groups). We feel that the key to the proof of this statement lies in a stronger version of Proposition 2.

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