SMOOTH EXTENSIONS OF LIPSCHITZIAN REAL FUNCTIONS

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Dedicated to Professor F. Guglielmino, with my deepest admiration, on his sixtieth birthday

ABSTRACT. In this short note we point out that any Lipschitzian real function f defined in a subset K of a Banach space E , with $\overline{\text{span}}(K) \neq E$, can be extended to a surjective, open and Lipschitzian real function g on E in such a way that, for every $r \in \mathbf{R}$, the set $g^{-1}(r)$ is arcwise connected. In fact, a more refined result is proved.

The aim of this short note is to point out Theorem 1 below. Before giving its statement, we recall that a topological space Y is said to be an absolute extensor for paracompact spaces (see [3]) if, for every paracompact topological space T , every closed set $A \subseteq T$ and every continuous function $\psi: A \rightarrow Y$, there exists a continuous function $\varphi: T \rightarrow Y$ such that $\varphi|_A = \psi$. We also recall that if U, V are two nonempty sets in a normed space $(E, \|\cdot\|)$, their Hausdorff distance, $d_H(U, V)$, is defined by putting

$$d_H(U, V) = \max \left\{ \sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\| \right\}.$$

Finally, if E is as above, any set $S \subseteq E$ will be regarded as a topological space endowed with the relative norm topology.

THEOREM 1. *Let $(E, \|\cdot\|)$ be a real Banach space, W a closed linear subspace of E , with $W \neq E$, K a nonempty subset of W and f a Lipschitzian real function on K , with Lipschitz constant L . Then, for every $M > 2L$, there exists a real function g on E with the following properties:*

- (a) $g(x) = f(x)$ for all $x \in K$;
- (b) g is Lipschitzian, with Lipschitz constant $(M + 2L)/2$;
- (c) for every $r \in \mathbf{R}$, the set $g^{-1}(r)$ is a nonempty absolute extensor for paracompact spaces;
- (d) for every $s, t \in \mathbf{R}$, one has

$$d_H(g^{-1}(s), g^{-1}(t)) \leq (2/(M - 2L))|s - t|.$$

PROOF. By Theorem I of [1], we can extend f to a Lipschitzian real function Ψ on E , with Lipschitz constant L . Moreover, by a corollary of the Hahn-Banach theorem, there exists a non-null continuous linear functional Λ on E such that $\Lambda(x) = 0$ for all $x \in W$. Now, for every $x \in E$, put $\Phi(x) = (M/2\|\Lambda\|_{E^*})\Lambda(x)$ as

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well as $g(x) = \Phi(x) + \Psi(x)$. The real function g , of course, satisfies (a) and (b). Let us show that g satisfies also (c) and (d). To this end, observe that, for every $u, v \in \mathbf{R}$, one has

$$(1) \quad d_H(\Phi^{-1}(u), \Phi^{-1}(v)) = (2/M)|u - v|.$$

Indeed, (1) follows easily from a classical result by G. Ascoli (see Lemma 1.2 of [5]), taking into account that $\|\Phi\|_{E^*} = M/2$. For every $r \in \mathbf{R}$ and $x \in E$, put $F_r(x) = \Phi^{-1}(r - \Psi(x))$ as well as $\text{Fix}(F_r) = \{y \in E: y \in F_r(y)\}$. Fix $r \in \mathbf{R}$. Plainly, we have

$$(2) \quad g^{-1}(r) = \text{Fix}(F_r).$$

By (1), for every $x, y \in E$, we obtain

$$(3) \quad d_H(F_r(x), F_r(y)) = (2/M)|\Psi(x) - \Psi(y)| \leq (2L/M)\|x - y\|.$$

Thus, since $2L/M < 1$, we infer that $x \rightarrow F_r(x)$ is a multivalued contraction from E into E , with closed convex values. Now, (c) is a direct consequence of (2) and of Theorem 1 of [4]. Finally, fix $s, t \in \mathbf{R}$. Taking into account (1), (2), (3) and Lemma 1 of [2], we then obtain

$$\begin{aligned} d_H(g^{-1}(s), g^{-1}(t)) &= d_H(\text{Fix}(F_s), \text{Fix}(F_t)) \\ &\leq \frac{M}{M - 2L} \sup_{x \in E} d_H(F_s(x), F_t(x)) = \frac{2}{M - 2L}|s - t|. \end{aligned}$$

This completes the proof.

REMARK. It is worth noting that, in Theorem 1, (d) implies that the real function g is open. Furthermore, (c) implies that each set $g^{-1}(r)$ is arcwise connected. Lemma 3.4 of [6] then ensures that the set $g^{-1}(A)$ is connected for every interval A of $\mathbf{R}$.

REFERENCES

1. J. Czipszér and L. Gehér, *Extensions of functions satisfying a Lipschitz condition*, Acta Math. Acad. Sci. Hungar. **6** (1955), 213–220.
2. T.-C. Lim, *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl. **110** (1985), 436–441.
3. E. Michael, *Some extension theorems for continuous functions*, Pacific J. Math. **3** (1953), 789–806.
4. B. Ricceri, *Une propriété topologique de l'ensemble des points fixes d'une contraction multivoque à valeurs convexes*, Rend. Accad. Naz. Lincei **81** (1987), 283–286.
5. I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, Berlin and New York, 1970.
6. R. E. Smithson, *Multifunctions*, Nieuw Arch. Wisk. **20** (1972), 32–53.

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