ALMOST JORDAN RINGS

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ABSTRACT. It is well known that any Jordan ring satisfies the identity: $2((ax)x)x + a((xx)x) = 3(a(xx))x$. We show that this identity along with commutativity implies the Jordan identity in any semiprime ring. The proof requires characteristic $\neq 2, 3$.

Introduction. A Jordan algebra is a commutative (nonassociative) algebra which satisfies the additional identity $((aa)x)a = (aa)(xa)$. This identity is called the "Jordan identity". It is not an irreducible identity with respect to commutativity (in the sense of Osborn [3, p. 184]), but is equivalent to the assumption of the following two identities, both of which are irreducible [2, p. 1114].

- (1) $((aa)a)a = (aa)(aa)$ fourth power associativity,
- (2) $2((ax)x)x + a((xx)x) = 3(a(xx))x$.

Under certain conditions, one of these identities alone is capable of implying the other. For example, Oehmke in [1] studied equation (1). Osborn in [2] and Petersson [4] studied equation (2). These papers relied on idempotent decompositions and arguments involving trace. This required assumptions like finite dimensionality and characteristic zero. Under rather restrictive conditions, it is known that simple commutative rings satisfying equation (1) are associative [1, p. 326], and those satisfying equation (2) are Jordan [2, Theorem 1].

In this paper we show that semiprime commutative rings satisfying equation (2) are Jordan. It is surprisingly uncomplicated. We will use the associator notation: $(x, y, z) = (xy)z - x(yz)$. We show that there is an ideal which squares to zero which contains all associators of the form (xx, a, x) . In a semiprime ring, this ideal must be zero. This implies that the ring satisfies the Jordan identity. Throughout this paper R will be a commutative ring of characteristic $\neq 2, 3$. We will often use a dot to indicate multiplication as well as juxtaposition. When this happens, it is intended that the product indicated by a dot will be done last. Thus by $(ab)(cd.e).f$ we mean $((ab)((cd)e))f$. We also use the Smiley notation for right multiplication: a prime on an element represents right multiplication by that element. Given a nonassociative ring $(R, +, *)$, the endomorphisms of $(R, +)$ form an associative algebra. For any a in R , right multiplication by a is an endomorphism and is indicated by a' . For

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elements a, b, c, d, e, f in R , we could consider the element $a'(bc)'(de \cdot f)'$ which is an endomorphism. When it acts on an element x in R , it sends x to $((xa)(bc))(de \cdot f)$. Linearization:

In an algebra over a field, Osborn [3, p. 177, Proposition 3.1] showed that to linearize an identity of degree n , there must be at least $n + 1$ distinct elements in the field. By the distributive laws, any ring is an algebra over the integers. The same type of argument will work when the scalars come from the integers as when they come from a field, except that we cannot necessarily get rid of the coefficients.

We do not wish to assume that our ring is an algebra over a field. If the scalars were a field, then we could cancel off any nonzero coefficient. Instead, we will only assume that we can cancel off coefficients of 2 and 3. We assume characteristic $\neq 2, 3$. By this we mean that in our rings:

$$nx = 0 \Rightarrow x = 0 \quad \text{for } n = 2 \text{ or } n = 3.$$

Suppose that $g(x_1, x_2, \dots, x_n)$ is a function of n arguments and is linear in all of them. Furthermore, assume that S is a set which is closed under addition and that $g(x, x, \dots, x)$ is in S for any x in R . We wish to argue that when we linearize the function g , each of its homogeneous components will be in S as well. If we replace x by $a + \delta b$ in g and use linearity to collect the homogeneous components, we get an identity of this form.

$$K_0 + \delta K_1 + \delta^2 K_2 + \delta^3 K_3 + \dots + \delta^n K_n = s_\delta \in S.$$

In this identity δ can be any integer. K_i represents the homogeneous component with i arguments " b " and $n - i$ arguments " a ". Written in matrix notation for different values of δ , we get a coefficient matrix which is a Vandermonde matrix.

$$\begin{bmatrix} 1 & \delta_0 & \delta_0^2 & \delta_0^3 & \dots & \delta_0^n \\ 1 & \delta_1 & \delta_1^2 & \delta_1^3 & \dots & \delta_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \delta_n & \delta_n^2 & \delta_n^3 & \dots & \delta_n^n \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_n \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix}.$$

Call this coefficient matrix M . Multiplying the equation by the adjoint of M will produce the result that $\det(M) * K_i \in S$ for $0 \leq i \leq n$.

Letting $\delta_0 = 0, \delta_1 = 1, \delta_2 = -1, \delta_3 = 2, \dots, \delta_{2k} = -k, \delta_{2k+1} = k + 1$, then $\det(M) = \prod_{i=1}^n i!$ where n is the degree of the function g . We shall only need $\det(M)$ for $n = 3, 4, 5$. The values of $\det(M)$ for these sizes are $12 = 2^2 3$, $288 = 2^5 3^2$, $34560 = 2^8 3^5$. We state these remarks in the following lemma.

LEMMA 1. *Let S be a subset of R which is closed under addition, and let g be a function of n arguments which is linear in each argument. Furthermore, suppose that $g(x, x, \dots, x) \in S$ for all x in R . If g is linearized, and its homogeneous parts are $K_0, K_1, K_2, \dots, K_n$, then $t * K_i \in S$ for $0 \leq i \leq n$ and $t = \prod_{i=1}^n i!$.*

Proof of the main theorem. We prove a series of six lemmas and then the main theorem.

LEMMA 2 (OSBORN).

$$f(x, y, z) = x'y'z' + z'y'x' - (xy)'z' - (yz)'x' - (zx)'y' + (xz \cdot y)' = 0.$$

PROOF. We linearize equation (2) to get the following.

$$\begin{aligned} & 2(ay \cdot x)x + a(yx \cdot x) - 3(a \cdot yx)x \\ & + 2(ax \cdot y)x + a(xy \cdot x) - 3(a \cdot xy)x = 0. \\ & + 2(ax \cdot x)y + a(xx \cdot y) - 3(a \cdot xx)y \end{aligned}$$

We interchange a and y and multiply by minus one to obtain

$$\begin{aligned} & -2(ya \cdot x)x - y(ax \cdot x) + 3(y \cdot ax)x \\ & - 2(yx \cdot a)x - y(xa \cdot x) + 3(y \cdot xa)x = 0. \\ & - 2(yx \cdot x)a - y(xx \cdot a) + 3(y \cdot xx)a \end{aligned}$$

Adding the above equations gives $4(a, xx, y) = 8(a, x, y)x$. Characteristic $\neq 2$ gives $(a, xx, y) = 2(a, x, y)x$. This linearizes to $2(a, xz, y) = 2(a, x, y)z + 2(a, z, y)x$. We thus get the derivation law

$$(3) \quad (a, xz, y) = (a, x, y)z + (a, z, y)x.$$

Expanding equation (3) and writing the elements as acting on 'a' gives the desired result stated in the lemma.

We will let I represent the additive span of all elements of the form (xx, x, x) . Eventually, we will show that both I and IR are in the middle nucleus, but for now, all we know about I is that it is closed under addition. We will let J be the additive span of all elements of the form (xx, y, x) . Clearly $I \subset J$ and R will be Jordan if and only if $J = 0$.

We now prove the following equation.

$$(4) \quad 3(xx, a, x) - 2(ax, x, x) - (xx, x, a) = 0.$$

PROOF. Expand the associators and the left-hand side becomes $-2(ax \cdot x)x + 3(a \cdot xx)x - a(x \cdot xx)$ which is zero by equation (2).

LEMMA 3. $2^7 3^2 J \subset I$.

PROOF. For any elements a and x in R , we have the following.

$$\begin{aligned} 4(xx, a, x) &= (xx, a, x) + 3(xx, a, x) \quad \text{using equation (4)} \\ &= (xx, a, x) + 2(ax, x, x) + (xx, x, a) \\ &= (ax, x, x) + (xa, x, x) + (xx, a, x) + (xx, x, a). \end{aligned}$$

This final right-hand side is a linearized form of (xx, x, x) . By Lemma 1 if we multiply this linearized form by the coefficient $288 = 2^5 3^2$, then it will land in I . This finishes the proof of Lemma 3.

LEMMA 4. $2^{10} 3^4 IR \subset I$.

PROOF. For any elements a and x in R we have these four identities.

$$\text{From equation (4)} \quad 6(xx, a, x)x - 4(ax, x, x)x - 2(xx, x, a)x = 0,$$

$$\text{From equation (3)} \quad 6(xx, ax, x) - 6(xx, x, x)a - 6(xx, a, x)x = 0,$$

$$\text{From equation (3)} \quad -2(ax, xx, x) + 4(ax, x, x)x = 0,$$

$$\text{From equation (3)} \quad -(xx, xx, a) + 2(xx, x, a)x = 0.$$

Adding these gives $6(xx, x, x)a = 6(xx, ax, x) - 2(ax, xx, x) - (xx, xx, a)$. This last identity can be written as

$$6(xx, x, x)a = 6(xx, ax, x) - \{(ax, xx, x) + (xa, xx, x) + (xx, xx, a)\}.$$

Multiply by 12 and use Lemma 1 to get $72(xx, x, x)a \in J$. From Lemma 3 we get $2^{10} 3^4 (xx, x, x)a \in 2^7 3^2 J \subset I$.

LEMMA 5. *For any a in R , we have*

$$(5) \quad \begin{aligned} & -24(aa)'a'a' + 12(aa)'(aa)'a' + 32(aa \cdot a)'a'a' \\ & -4(aa)'(aa \cdot a)' - 8((aa \cdot a)a)'a' - 12(aa \cdot aa)'a' = 0. \\ & +2((aa \cdot a)a \cdot a)' - 3((aa \cdot aa)a)' + 5((aa \cdot a) \cdot aa)' \end{aligned}$$

PROOF. When the following linear combination of f identities is expanded, it gives equation (5).

$$\begin{aligned} & -8a'f(a, a, a)a' + 8f(a, a, a)a'a' - 12f(a, aa, a)a' \\ & + 4f(a, aa \cdot a, a) + ((aa)f(a, a, a))'. \end{aligned}$$

We will wish to use the linearized form of equation (5). Since equation (5) has five arguments, using Lemma 1 directly would require a coefficient with a factor of 5. Fortunately, we do not need to linearize at all. If we substitute one b and four a 's in each of the 5 possible ways in the proof, the individual terms remain zero since f is an identity for any substitution in its arguments. The sum of all these terms will be the linearized form of equation (5).

LEMMA 6. *R satisfies the following identity.*

$$(w, (aa, a, a), b) = 0 \quad \text{for all } w, a, b \text{ in } R.$$

PROOF. The proof is done in much the same way that Lemma 5 was proved. However, to leave all the expansions to the reader is too discouraging. We have done all the expansions and put them in Table I. To keep the size of the table manageable, we have adopted a shorthand for the notation. Each term consists of four a 's and one b . There are 5 different permutations of these letters. They are indicated by A, B, C, D, E . $A = baaaa$, $B = abaaa$, $C = aabaa$, $D = aaaba$, $E = aaaab$. Besides the permutation of the four a 's and 1 b , each term has an "association". There are 20 different ways that the terms can be associated. Terms of like association are listed in the same column. The association type represented by a particular column is given below the main part of the table. If an identity gave more than one term of a particular association type, they are listed one above the other in the appropriate column. The "." represents zero.

The far left-hand column consists of identities written using the " f " function. Each identity is expanded out and forms a new row in the matrix. Since each row of the matrix is an identity, the sum of all these rows is again an identity. The sum is written across the bottom. In the bottom, the x means $A + B + C + D + E$. Comparing the columns where the x 's are, one can see that the x 's represent a linearization of identity equation (5). In the paragraph following Lemma 5, we explained why the linearized form of equation (5) is still an identity. Since all the x 's cancel, we are left with $48(aa, b, a)a' - 48((aa, b, a)a)' = 0$. It follows that

$$(6) \quad (w, (aa, b, a), a) = 0 \quad \text{for all } a, b, w \text{ in } R.$$

Linearizing this on " a " using Lemma 1 and characteristic $\neq 2, 3$ gives

$$(w, (ya, b, a), a) + (w, (ay, b, a), a) + (w, (aa, b, y), a) + (w, (aa, b, a), y) = 0.$$

We now set $b = a$ to get

$$(w, (ya, a, a), a) + (w, (ay, a, a), a) + (w, (aa, a, y), a) + (w, (aa, a, a), y) = 0.$$

TABLE I

Type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$-8a'a'f(a, b, a)$	$-16D$	.	.	$16D$	.	.	.	.	.	.	$-8E$	.	.	.	.	.	.	.	.	.
				$+8E$																
$-8a'b'f(a, a, a)$	$-16B$	.	.	$24B$	.	.	.	.	.	.	$-8B$	.	.	.	.	.	.	.	.	.
$-8a'f(a, a, a)b'$	$-16E$	.	$24E$	.	.	.	.	.	.	.	$-8E$	.	.	.	.	.	.	.	.	.
$8f(a, a, a)a'b'$	$16E$	$-24E$	.	.	.	.	.	.	.	$8E$	.	.	.	.	.	.	.	.	.	.
$8f(a, a, a)b'a'$	$16D$	$-24D$	.	.	.	.	.	.	.	$8D$	.	.	.	.	.	.	.	.	.	.
$8f(a, b, a)a'a'$	$16B$	$-16B$	.	.	.	.	.	.	.	$8C$	.	.	.	.	.	.	.	.	.	.
		$-8C$																		
$-16f(aa, b, a)a'$	$-16C$	.	$-16B$	.	$16B$	.	.	.	$16C$	.	.	.	.	$-16D$	.	.	.	.	.	.
									$16D$											
$-32f(ab, a, a)a'$	.	$-32B$	.	$-32D$	.	$32D$	.	.	$64B$	.	.	.	.	$-32B$	.	.	.	.	.	.
$-8a'f(a, aa, b)$	.	.	.	$-8B$	.	.	.	$8C$	.	.	$8D$	.	.	.	.	.	$-8C$	.	.	.
				$-8E$							$8E$									
$8a'f(a, ab, a)$	.	.	.	$16D$	.	.	.	$-8E$	.	.	$-16C$	.	.	.	.	.	$8C$	.	.	.
$-12f(a, aa, a)b'$	.	.	$-24E$	.	.	$12E$	.	.	.	$24E$	.	.	.	.	.	$-12E$	.	.	.	.
$8f(aa \cdot a, b, a)$	.	.	.	.	.	.	.	.	.	$8D$	.	$8B$	$-8B$	$-8D$	.	.	.	$8E$	.	.
														$-8E$						
$8f(aa \cdot b, a, a)$	.	.	.	.	.	.	.	.	.	$8C$	.	$8E$	$-8E$	$-16C$	.	.	.	$8C$	.	.
$-8f(aa, ab, a)$	.	.	.	.	.	.	$-8D$	$-8C$	.	.	.	.	.	$8B$	.	$8D$	.	.	.	$-8E$
														$+8E$						
$8f(ab, aa, a)$	.	.	.	.	.	.	$8B$	$8E$	.	.	.	.	.	$-8B$	.	$-8D$	.	.	.	$8B$
														$-8E$						
$-4f(a, aa \cdot b, a)$	.	.	.	.	.	.	.	.	.	$-8D$	.	$4E$	.	$8C$	.	.	.	.	.	$-4C$
$8f(a, ab \cdot a, a)$	.	.	.	.	.	.	.	.	.	$16C$	.	$-8D$	.	$-16B$	.	.	.	.	.	$8B$
$16[bf(a, a, a)]'a'$	.	.	.	.	.	.	.	.	.	.	.	.	.	$32A$	.	.	.	.	.	.
														$-48C$						
														$+16D$						
$-16[bf(a, a, a) \cdot a]'$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$-32A$	.	.	.
																	$+48C$			
																	$-16D$			
$-6[bf(aa, a, a)]'$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$-6C$	.	$-6A$	.
																	$+12D$		$+6C$	
																	$-6E$			
$-3[bf(a, aa, a)]'$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$6D$	$-6C$	$3C$	
																	$-3E$			
$6[af(a, ab, a)]'$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$-12A$	$18C$	$-6E$	
$24[ab \cdot f(a, a, a)]'$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$48B$	$-72B$	$24D$	
0 =	.	$-24x$	.	.	.	$12x$	.	.	$32x$	.	.	$-4x$	.	$-8x$	$-12x$	.	.	$2x$	$-3x$	$5x$
														$-48C$	$48C$		$48C$	$-48B$		

Types used in this table are:

1. $R'R'R'R'R'$	2. $(RR)'R'R'R'$	3. $R'(RR)'R'R'$	4. $R'R'(RR)'R'$
5. $R'R'R'(RR)'$	6. $(RR)'(RR)'R'$	7. $R'(RR)'(RR)'$	8. $(RR)'R'(RR)'$
9. $(RR \cdot R)'R'R'$	10. $R'(RR \cdot R)'R'$	11. $R'R'(RR \cdot R)'$	12. $(RR)'(RR \cdot R)'$
13. $(RR \cdot R)'(RR)'$	14. $((RR \cdot R)R)'R'$	15. $(RR \cdot RR)'R'$	16. $R'((RR \cdot R)R)'$
17. $R'(RR \cdot RR)'$	18. $((RR \cdot R)R \cdot R)'$	19. $((RR \cdot RR)R)'$	20. $((RR \cdot R) \cdot RR)'$

Then by equation (4) we get

$$3(w, (aa, y, a), a) + (w, (aa, a, a), y) = 0.$$

Using equation (6) again, we get

$$(w, (aa, a, a), b) = 0.$$

LEMMA 7. *I and IR are contained in the middle nucleus.*

PROOF. Lemma 6 gives $(R, I, R) = 0$ so I is in the middle nucleus. Lemma 4 gives $2^{10}3^4IR \subset I$. Thus $2^{10}3^4(R, IR, R) = 0$. Characteristic $\neq 2, 3$ forces $(R, IR, R) = 0$.

THEOREM. *Let R be a commutative ring satisfying equation (2). If R is semi-prime and of characteristic $\neq 2, 3$, then R is a Jordan ring.*

PROOF. Let $T = \{x \in R | x(R, R, R) = (R, x, R) = 0\}$. T is clearly closed under addition. Furthermore, if $x \in T$ and $y \in R$, then $xy \cdot (R, R, R) = yx \cdot (R, R, R) \subset y \cdot x(R, R, R) + (y, x, (R, R, R)) = 0$ by definition of T . Using equation (3), we get $(R, xy, R) \subset (R, x, R)y + (R, y, R)x = 0$ by definition of T . We have shown that T is an ideal. Let $T' = \{x \in R | xT = (R, x, T) = 0\}$. T' is closed under addition. Let $x \in T'$ and $y \in R$. Then $xy \cdot T \subset yx \cdot T \subset y \cdot xT + (y, x, T) = 0$ by definition of T' . Also, using equation (3), $(R, xy, T) \subset (R, x, T)y + (R, y, T)x \subset 0 + Tx = 0$ using the definition of T' and the fact that T is an ideal. We have shown that T' is an ideal. Now, $T \cap T'$ is a trivial ideal and must be zero.

By equation (3) and Lemma 7, $I(R, R, R) \subset (R, IR, R) + (R, I, R)R = 0$. This and Lemma 7 prove that $I \subset T$. Since $I \subset (R, R, R)$, $IT = 0$ by definition of T . Since I is in the middle nucleus, $(R, I, T) = 0$. Thus $I \subset T'$. We conclude that $I = 0$. By Lemma 3, $2^73^2J = 0$. Characteristic $\neq 2, 3$ proves that $J = 0$. Therefore, the ring A is Jordan.

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