M_n AS A 0,1-SUBLATTICE OF $\text{Con } \mathbf{A}$ DOES NOT FORCE THE TERM CONDITION

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ABSTRACT. For every $n \geq 3$ there exists a finite nonabelian algebra whose congruence lattice has M_n as a 0,1-sublattice. This answers a question of R. McKenzie and D. Hobby.

DEFINITION 0.1. Suppose L and L_1 are bounded lattices. A *copy* of L (in L_1) is any sublattice L' of L_1 which is isomorphic to L . L' is a *0,1-copy* of L if it includes the least and greatest elements of L_1 ; in this case L' is also called a *0,1-sublattice* of L_1 .

DEFINITION 0.2. For $n \geq 1$, M_n is the finite lattice of height 2 having exactly n atoms. For example, M_6 is

Suppose G is a group and $N(G)$ is its lattice of normal subgroups. There is a trivial proof, using the commutator operation on normal subgroups, that if $N(G)$ has a 0,1-copy of M_3 then G is abelian. This same proof extends, via the general commutator theory of universal algebra, to any algebra $\mathbf{A}$ belonging to a variety whose congruence lattices satisfy the modular law. Here $N(G)$ is replaced by $\text{Con } \mathbf{A}$ (the lattice of congruence relations of $\mathbf{A}$), while ‘abelian’ means the following ‘term condition’:

DEFINITION 0.3. An algebra $\mathbf{A}$ is *abelian* if for every $n \geq 1$, every $(n+1)$ -ary term $t(x, y_1, \dots, y_n)$ in the language of $\mathbf{A}$, and all $a, b, c_1, \dots, c_n, d_1, \dots, d_n \in A$,

$$t^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(a, \vec{d}) \quad \text{iff} \quad t^{\mathbf{A}}(b, \vec{c}) = t^{\mathbf{A}}(b, \vec{d}).$$

In their forthcoming book on tame congruence theory [1], R. McKenzie and D. Hobby ask whether the above phenomenon— $\text{Con } \mathbf{A}$ having M_3 as a 0,1-sublattice

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forcing $\mathbf{A}$ to be abelian—holds for all *finite* algebras (without assuming congruence modularity); and if not, whether there is any $n > 3$ for which having M_n as a 0, 1-sublattice suffices. The purpose of this paper is to note that the answer to the first question is positive (without requiring finiteness) in case $\mathbf{A}$ belongs to a locally finite variety which ‘omits type 1’, while the answers to both questions are negative in general.

1. In this section I prove the positive answer for algebras belonging to locally finite varieties which omit type 1. Definition 1.1 is taken from [2], and Lemma 1.4 is a slight improvement of Lemma 4.153 found there.

DEFINITION 1.1. Suppose α, β, δ are congruences of an algebra $\mathbf{A}$.

(1) α *centralizes* β *modulo* δ if, for every $n \geq 1$, every term $t(x, y_1, \dots, y_n)$ in the language of $\mathbf{A}$, and all $\langle a, b \rangle \in \alpha$, $\langle c_1, d_1 \rangle, \dots, \langle c_n, d_n \rangle \in \beta$,

$$t^{\mathbf{A}}(a, \vec{c}) \stackrel{\delta}{\equiv} t^{\mathbf{A}}(a, \vec{d}) \quad \text{iff} \quad t^{\mathbf{A}}(b, \vec{c}) \stackrel{\delta}{\equiv} t^{\mathbf{A}}(b, \vec{d}).$$

(2) If $\alpha \geq \delta$, then α is *abelian over* δ if α centralizes itself modulo δ . (Thus $\mathbf{A}$ is abelian iff $\nabla_{\mathbf{A}}$ is abelian over $\Delta_{\mathbf{A}}$.)

DEFINITION 1.2. Suppose δ, θ are congruences of $\mathbf{A}$ with $\theta \geq \delta$.

(1) θ is *solvable over* δ if there exist $\alpha_0, \dots, \alpha_k \in \text{Con } \mathbf{A}$ such that $\alpha_0 = \delta$, $\alpha_k = \theta$, and α_{i+1} is abelian over α_i for each $i < k$.

(2) θ is *locally solvable over* δ if in every finitely generated subalgebra $\mathbf{B} \leq \mathbf{A}$, $\theta \cap B^2$ is solvable over $\delta \cap B^2$.

LEMMA 1.3. Suppose $\mathbf{A}$ is an algebra and $\text{Con } \mathbf{A}$ has a copy of M_3 , as in the picture below.

If $\alpha_2 \circ \alpha_3 \circ \alpha_2 = \theta$, then α_1 centralizes θ modulo α_2 , and hence modulo δ .

PROOF. Let $t(x, y_1, \dots, y_n)$ be a term in the language of $\mathbf{A}$, and $\langle a, b \rangle \in \alpha_1$, $\langle c_1, d_1 \rangle, \dots, \langle c_n, d_n \rangle \in \theta$. Pick c'_i, d'_i such that $c_i \stackrel{\alpha_2}{\equiv} c'_i \stackrel{\alpha_3}{\equiv} d'_i \stackrel{\alpha_2}{\equiv} d_i$ for each $i = 1, \dots, n$. Now suppose $t^{\mathbf{A}}(a, \vec{c}) \stackrel{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d})$. Then

$$t^{\mathbf{A}}(a, \vec{c}') \stackrel{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{c}) \stackrel{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d}) \stackrel{\alpha_2}{\equiv} t^{\mathbf{A}}(a, \vec{d}') \quad \text{and} \quad t^{\mathbf{A}}(a, \vec{c}') \stackrel{\alpha_3}{\equiv} t^{\mathbf{A}}(a, \vec{d}'),$$

so

$$\langle t^{\mathbf{A}}(a, \vec{c}'), t^{\mathbf{A}}(a, \vec{d}') \rangle \in \alpha_2 \cap \alpha_3 \subseteq \alpha_1.$$

Similarly, $\langle t^{\mathbf{A}}(b, \vec{c}'), t^{\mathbf{A}}(b, \vec{d}') \rangle \in \alpha_1 \cap \alpha_3 \subseteq \alpha_2$. So $t^{\mathbf{A}}(b, \vec{c}) \stackrel{\alpha_2}{\equiv} t^{\mathbf{A}}(b, \vec{d})$, which proves that α_1 centralizes θ modulo α_2 .

Now α_1 automatically centralizes θ modulo $\alpha_1 \cap \theta = \alpha_1$. It follows that α_1 centralizes θ modulo $\alpha_2 \cap \alpha_1 = \delta$. $\square$

COROLLARY 1.4. *Suppose $\text{Con } \mathbf{A}$ has a copy of M_3 as in the previous lemma. If for some $\{i_1, i_2\} = \{2, 3\}$ and $\{j_1, j_2\} = \{1, 3\}$ the condition*

$$\alpha_{i_1} \circ \alpha_{i_2} \circ \alpha_{i_1} = \alpha_{j_1} \circ \alpha_{j_2} \circ \alpha_{j_1} = \theta$$

is met, then θ is abelian over δ .

PROOF. By the previous lemma, α_1 and α_2 both centralize θ modulo δ . It follows that $\alpha_1 \vee \alpha_2 = \theta$ centralizes θ modulo δ . $\square$

The concept of a locally finite variety V omitting type 1 is central to tame congruence theory. One way to define this concept is as follows: V *omits type 1* if for every $\mathbf{A} \in V$ and every $\theta \in \text{Con } \mathbf{A}$ different from Δ_A , there is an $(n+1)$ -ary term $t(x, \vec{y})$ in the language of V , and there exist elements $a \stackrel{\theta}{=} b$, $c_i \stackrel{\theta}{=} d_i \stackrel{\theta}{=} e_i$ ($1 \leq i \leq n$) of A , such that $t^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(b, \vec{d})$ but $t^{\mathbf{A}}(a, \vec{e}) \neq t^{\mathbf{A}}(b, \vec{e})$. Most interesting locally finite varieties, including those whose congruence lattices satisfy some nontrivial lattice identity, omit type 1. Omitting type 1 is equivalent to the variety satisfying some nontrivial idempotent Mal'cev condition, and also to the permuting of congruences in locally solvable intervals. For more information, the reader is referred to [1, Chapters 1–9].

THEOREM 1.5. *Suppose $\mathbf{A}$ belongs to a locally finite variety which omits type 1, and $\text{Con } \mathbf{A}$ has a copy of M_3 as in Lemma 1.3. Then θ is abelian over δ . If $\text{Con } \mathbf{A}$ has a 0, 1-copy of M_3 , then $\mathbf{A}$ is abelian.*

PROOF. It is an easy exercise to show that θ centralizes each α_i modulo δ , and hence each α_i is abelian over δ . It follows that each α_i is locally solvable over δ ; hence [1, Lemma 7.4 and Corollary 7.5] $\theta = \bigvee_i \alpha_i$ is locally solvable over δ . Since $\mathbf{A}$ belongs to a locally finite variety which omits type 1, congruences in the interval $[\delta, \theta]$ permute [1, Theorem 7.12]). So Corollary 1.4 applies. $\square$

2. In this section I construct, for every prime p , a finite nonabelian algebra whose congruence lattice has a 0, 1-copy of M_{p+1} . In what follows, let p be a fixed prime.

DEFINITION 2.1. (1) $\mathcal{L}_p$ is the relational language containing the binary relation symbols $\theta_0, \theta_1, \dots, \theta_p$.

(2) An M_{p+1} -model is a finite $\mathcal{L}_p$ -structure $\mathbf{A} = \langle A; \theta_0^{\mathbf{A}}, \dots, \theta_p^{\mathbf{A}} \rangle$ which satisfies:

(i) $|A| > 1$.

(ii) Each $\theta_i^{\mathbf{A}}$ is an equivalence relation on A .

(iii) $i \neq j$ implies $\theta_i^{\mathbf{A}} \vee \theta_j^{\mathbf{A}} = \nabla_A$ and $\theta_i^{\mathbf{A}} \cap \theta_j^{\mathbf{A}} = \Delta_A$.

Note that if $\mathbf{A}$ is an M_{p+1} -model and h is a homomorphism $h: \mathbf{A}^n \rightarrow \mathbf{A}$ for some $n \geq 1$, then $\langle A; h \rangle$ is a finite algebra whose congruence lattice has a 0, 1-copy of M_{p+1} .

DEFINITION 2.2. (1) The equivalence relations $\Theta_0, \dots, \Theta_p$ on $\mathbf{Z}^2$ are defined by

$$\Theta_i = \{ \langle (x, y), (x', y') \rangle : y' - y \equiv i(x' - x) \pmod{p} \} \quad \text{for } 0 \leq i < p,$$

$$\Theta_p = \{ \langle (x, y), (x', y') \rangle : x \equiv x' \pmod{p} \}.$$

(2) An M_{p+1} -model $\mathbf{A}$ is *standard* if $A \subseteq \mathbf{Z}^2$ and $\theta_i^{\mathbf{A}} \subseteq \Theta_i$ for each $i = 0, \dots, p$; i.e. if the inclusion map $A \xrightarrow{\text{incl}} \mathbf{Z}^2$ is an injective homomorphism from $\mathbf{A}$ to $\langle \mathbf{Z}^2; \Theta_0, \dots, \Theta_p \rangle$.

EXAMPLE 2.3. Define the $\mathcal{L}_p$ -structure $\mathbf{V}$ by $V = \{0, 1, \dots, p-1\}^2$ and $\theta_i^{\mathbf{V}} = \Theta_i \cap V^2$. Then $\mathbf{V}$ is a standard M_{p+1} -model. It corresponds to the congruence lattice of the 2-dimensional vector space over $GF(p)$.

Standard M_{p+1} -models can be visualized as labelled graphs in the plane. They can also be transformed and combined to make new standard M_{p+1} -models.

DEFINITION 2.4. Let $\mathbf{A}, \mathbf{B}$ be standard M_{p+1} -models, $v \in \mathbf{Z}^2$, and $\lambda \in \mathbf{Z} \setminus \{0\}$.

(1) The *translation of $\mathbf{A}$ by v* , denoted $\mathbf{A} + v$, is the $\mathcal{L}_p$ -structure $\mathbf{A}'$ given by

$$\begin{aligned} A' &= \{a + v : a \in A\}, \\ \theta_i^{\mathbf{A}'} &= \{\langle a + v, a' + v \rangle : \langle a, a' \rangle \in \theta_i^{\mathbf{A}}\}. \end{aligned}$$

(2) The *dilation of $\mathbf{A}$ by λ* , denoted $\lambda\mathbf{A}$, is the $\mathcal{L}_p$ -structure $\mathbf{A}'$ given by

$$\begin{aligned} A' &= \{\lambda a : a \in A\}, \\ \theta_i^{\mathbf{A}'} &= \{\langle \lambda a, \lambda a' \rangle : \langle a, a' \rangle \in \theta_i^{\mathbf{A}}\}. \end{aligned}$$

(3) Suppose $|A \cap B| = 1$. The *one point union* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cup \mathbf{B}$, is the $\mathcal{L}_p$ -structure $\mathbf{A}'$ where $A' = A \cup B$ and each $\theta_i^{\mathbf{A}'}$ is the transitive closure of $\theta_i^{\mathbf{A}} \cup \theta_i^{\mathbf{B}}$.

(4) Suppose $u \in A$, $v \notin A$ and $\langle u, v \rangle \in \Theta_0 \cap \Theta_p$. The *perturbation of $\mathbf{A}$ sending u to v* is the $\mathcal{L}_p$ -structure $\mathbf{A}'$ whose universe is $A' = (A \setminus \{u\}) \cup \{v\}$, and which is isomorphic to $\mathbf{A}$ via the isomorphism $f : \mathbf{A} \rightarrow \mathbf{A}'$ defined by

$$f(a) = \begin{cases} a & \text{if } a \neq u, \\ v & \text{if } a = u. \end{cases}$$

LEMMA 2.5. *All translations, dilations, one point unions and perturbations of standard M_{p+1} -models are standard M_{p+1} -models.* $\square$

The next construction is rather more complicated. In what follows, π_i ($i = 1, 2$) are the projections of $\mathbf{Z}^2$ onto $\mathbf{Z}$.

DEFINITION 2.6. Let $\mathbf{A}$ be a standard M_{p+1} -model.

(1) The numbers $\underline{m}_1, \overline{m}_1, \underline{m}_2, \overline{m}_2 \in \mathbf{Z}$ are defined to be $\underline{m}_i = \min(\pi_i(A))$ and $\overline{m}_i = \max(\pi_i(A))$, $i = 1, 2$.

(2) The *lower-left* and *upper-right corners* of $\mathbf{A}$ are the points $\underline{m} = (\underline{m}_1, \underline{m}_2)$ and $\overline{m} = (\overline{m}_1, \overline{m}_2)$.

(3) The *diameter* of $\mathbf{A}$ is $\max(\overline{m}_1 - \underline{m}_1, \overline{m}_2 - \underline{m}_2)$.

(4) The *gauge* of $\mathbf{A}$ is

$$\min(\{|u - v| : u, v \in \pi_1(A), u \neq v\} \cup \{|u - v| : u, v \in \pi_2(A), u \neq v\}).$$

DEFINITION 2.7. Let $\mathbf{A}, \mathbf{B}$ be standard M_{p+1} -models. A *fiddle of $\mathbf{B}$ by $\mathbf{A}$* consists of a family of standard M_{p+1} -models $\mathbf{B}_a$ and surjective homomorphisms $f_a : \mathbf{B} \rightarrow \mathbf{B}_a$, indexed by A , which satisfies the following:

(i) For each $i = 0, \dots, p$ and each $\theta_i^{\mathbf{A}}$ equivalence class N there exists an equivalence relation θ_i^N on B such that for all $a \in N$ and $b, b' \in B$,

$$\langle f_a(b), f_a(b') \rangle \in \theta_i^{\mathbf{B}_a} \quad \text{iff} \quad \langle b, b' \rangle \in \theta_i^N.$$

(ii) For all $a \in A$ and $b \in B$, $\langle f_a(b), b \rangle \in \Theta_0 \cap \Theta_p$.

DEFINITION 2.8. Let $\mathbf{A}, \mathbf{B}$ be standard M_{p+1} -models and suppose $\bar{f} = (\mathbf{B}_a, f_a)_{a \in A}$ is a fiddle of $\mathbf{B}$ by $\mathbf{A}$ which satisfies:

- (i) The lower-left corner of each $\mathbf{B}_a$ is $\bar{0} = (0, 0)$.
- (ii) The gauge of $\mathbf{A}$ is greater than the diameter of each $\mathbf{B}_a$.

Then the *fiddled product* $\mathbf{A} \times_{\bar{f}} \mathbf{B}$ is the $\mathcal{L}_p$ -structure $\mathbf{C}$ where $C = \bigcup \{B_a + a : a \in A\}$ and $\theta_i^{\mathbf{C}}$ is the transitive closure of

$$\bigcup \{ \theta_i^{\mathbf{B}_a + a} : a \in A \} \cup \{ \langle f_a(b) + a, f_{a'}(b) + a' \rangle : \langle a, a' \rangle \in \theta_i^{\mathbf{A}}, b \in B \}$$

LEMMA 2.9. Every fiddled product of standard M_{p+1} -models is a standard M_{p+1} -model.

PROOF. Let $\mathbf{A}, \mathbf{B}$ and $\bar{f} = (\mathbf{B}_a, f_a)_{a \in A}$ be as in Definition 2.8, and let $\mathbf{C} = \mathbf{A} \times_{\bar{f}} \mathbf{B}$. Clearly $\theta_i^{\mathbf{C}} \vee \theta_j^{\mathbf{C}} = \nabla_{\mathbf{C}}$ for $i \neq j$. For each $i = 0, \dots, p$ and each $\theta_i^{\mathbf{A}}$ equivalence class N , let θ_i^N be as in Definition 2.7. The conditions of Definition 2.8 imply that each $c \in C$ has a unique representation as $a + b$ for some $a \in A$ and $b \in B_a$. This fact and condition (i) of Definition 2.7 imply that for any $a, a' \in A$ and $b, b' \in B$, and any i ,

$$\langle f_a(b) + a, f_{a'}(b') + a' \rangle \in \theta_i^{\mathbf{C}} \quad \text{iff} \quad \langle a, a' \rangle \in \theta_i^{\mathbf{A}} \text{ and } \langle b, b' \rangle \in \theta_i^N \text{ where } N = a/\theta_i^{\mathbf{A}}.$$

This implies that $\theta_i^{\mathbf{C}} \cap \theta_j^{\mathbf{C}} = \Delta_{\mathbf{C}}$ for $i \neq j$, so $\mathbf{C}$ is an M_{p+1} -model. Condition (ii) of Definition 2.7 guarantees that $\mathbf{C}$ is standard. $\square$

The constructions defined above also provide an abundance of homomorphisms.

LEMMA 2.10. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be standard M_{p+1} -models.

- (1) For each $v \in \mathbf{Z}^2$, the map $a \mapsto a + v$ is an isomorphism from $\mathbf{A}$ to $\mathbf{A} + v$.
- (2) For each $\lambda \in \mathbf{Z} \setminus \{0\}$, the map $a \mapsto \lambda a$ is an isomorphism from $\mathbf{A}$ to $\lambda \mathbf{A}$.
- (3) Suppose $|A \cap B| = 1$. Then the inclusion map $B \hookrightarrow A \cup B$ is an embedding of $\mathbf{B}$ into $\mathbf{A} \cup \mathbf{B}$.
- (4) Suppose $A \cap B = A \cap C = \{a_0\}$ and $h: \mathbf{B} \rightarrow \mathbf{C}$ is a homomorphism such that $h(a_0) = a_0$. Then the map $\text{id}_A \cup h$ is a homomorphism from $\mathbf{A} \cup \mathbf{B}$ to $\mathbf{A} \cup \mathbf{C}$.
- (5) Suppose $\bar{f} = (\mathbf{B}_a, f_a)_{a \in A}$ is a fiddle of $\mathbf{B}$ by $\mathbf{A}$ which satisfies the conditions of Definition 2.8. Then:

- (i) The map $(a, b) \mapsto f_a(b) + a$ is a homomorphism from $\mathbf{A} \times \mathbf{B}$ to $\mathbf{A} \times_{\bar{f}} \mathbf{B}$.
- (ii) The map $a + b \mapsto a$, for $a \in A$ and $b \in B_a$, is a well-defined homomorphism from $\mathbf{A} \times_{\bar{f}} \mathbf{B}$ to $\mathbf{A}$.
- (6) Let $\mathbf{V}$ be the standard M_{p+1} -model given in Example 2.3. For each $x \in \mathbf{Z}$ let $\text{res}_p(x)$ denote the least residue of $x \bmod p$. Then the map $(x, y) \mapsto (\text{res}_p(x), \text{res}_p(y))$ is a homomorphism from $\langle \mathbf{Z}^2; \Theta_0, \dots, \Theta_p \rangle$ to $\mathbf{V}$.

PROOF. The only claim worth considering is 5(ii); it follows from the description of each $\theta_i^{\mathbf{C}}$ ($\mathbf{C} = \mathbf{A} \times_{\bar{f}} \mathbf{B}$) contained in the proof of Lemma 2.9. $\square$

THEOREM 2.11. Suppose there exists a standard M_{p+1} -model $\mathbf{A}$ and points $c, d \in A$ such that

- (i) For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c, d \rangle \notin \theta_i^{\mathbf{A}} \circ \theta_j^{\mathbf{A}} \circ \theta_i^{\mathbf{A}}$.
- (ii) $\langle c, d \rangle \in \Theta_0 \cap \Theta_p$.

Then there exists a finite nonabelian algebra whose congruence lattice contains a 0, 1-copy of M_{p+1} .

EXAMPLE. If $p = 2$, then the $\mathcal{L}_2$ -structure pictured below satisfies the hypotheses of the theorem. (θ_2^A and θ_0^A are the kernels of the first and second projections, respectively, while θ_1^A is defined by the diagonal edges.)

PROOF OF THEOREM 2.11. It is easy to verify that the hypotheses imply $|A| \geq 4$. Pick elements $a_0, b_0 \in A$ such that a_0, b_0, c, d are all distinct. Via perturbations of a_0 and b_0 , A can be fixed so that a_0 and b_0 are the lower-left and upper-right corners $\underline{m}$ and $\overline{m}$ of A . (These corners will be used in the formation of one point unions.) It can also be assumed that $\underline{m} = \overline{0}$ (by translating A if necessary).

Let V be the standard M_{p+1} -model of Example 2.3; note that its lower-left corner is $\overline{0}$, and $\overline{0} \in V$. Thus $B = A \cup (V + \overline{m})$ is defined, and B satisfies

(i)' For all $0 \leq i, j \leq p$ with $i \neq j$, $\langle c, d \rangle \notin \theta_i^B \circ \theta_j^B \circ \theta_i^B$.

Next define a fiddle of B by V . For each $i = 0, \dots, p$ let $\bar{\theta}_i$ be the symmetric transitive closure of $\theta_i^B \cup \{\langle c, d \rangle\}$. For each $v \in V \setminus \{\overline{0}\}$ pick the unique $i \in \{0, \dots, p\}$ such that $\langle v, \overline{0} \rangle \in \Theta_i$ and define B_v by

$$B_v = B, \\ \theta_j^{B_v} = \begin{cases} \theta_j^B & \text{if } j \neq i, \\ \bar{\theta}_i & \text{if } j = i. \end{cases}$$

Also let $f_v = \text{id}_B$. Finally, define $B_{\overline{0}}$ to be the $\mathcal{L}_p$ -structure where $B_{\overline{0}} = B \setminus \{d\}$ and each $\theta_i^{B_{\overline{0}}}$ is the symmetric transitive closure of

$$(\theta_i^B \cap (B_{\overline{0}})^2) \cup \{\langle b, c \rangle : b \neq d, \langle b, d \rangle \in \theta_i^B\}$$

Also define $f_{\overline{0}}: B \rightarrow B_{\overline{0}}$ by

$$f_{\overline{0}}(b) = \begin{cases} b & \text{if } b \neq d, \\ c & \text{if } b = d. \end{cases}$$

Property (i)' implies that each B_v is an M_{p+1} -model while hypothesis (ii) implies it is standard. Clearly each f_v is a surjective homomorphism from B to B_v , and

$\langle f_v(b), b \rangle \in \Theta_0 \cap \Theta_p$ for all $b \in B$. Suppose $i \in \{0, \dots, p\}$ and N is a $\theta_i^{\mathbf{V}}$ equivalence class. If $\bar{0} \notin N$, then the equivalence relation $\theta_i^N = \theta_i^{\mathbf{B}}$ satisfies condition (i) of Definition 2.7, while if $\bar{0} \in N$ then $\theta_i^N = \bar{\theta}_i$ satisfies the condition. Thus $\bar{f} = \langle \mathbf{B}_v, f_v \rangle_{v \in V}$ is a fiddle of $\mathbf{B}$ by $\mathbf{V}$.

$\bar{f}$ satisfies the first condition of Definition 2.8 but not the second. This is easily remedied. Let λ be an integer greater than the diameter of $\mathbf{B}$, and for each $v \in V$ let $\mathbf{B}'_{\lambda v} = \mathbf{B}_v$, $f'_{\lambda v} = f_v$ and $\bar{f}' = \langle \mathbf{B}'_{\lambda v}, f'_{\lambda v} \rangle_{\lambda v \in \lambda V}$. The gauge of $\lambda \mathbf{V}$ is λ , so $\lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B}$ is defined. Note that $\lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B}$ contains its lower-left corner, which is $\bar{0}$. Thus $\mathbf{C} = \mathbf{A} \cup [(\lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B}) + \bar{m}]$ is defined.

Now consider the following sequences of homomorphisms of $\mathcal{L}_p$ -structures. (Unnamed maps are the canonical homomorphisms defined in Lemma 2.10.)

$$(1) \quad \alpha: (\lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B}) + \bar{m} \cong \lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B} \rightarrow \lambda \mathbf{V} \cong \mathbf{V} \cong \mathbf{V} + \bar{m}.$$

Note that $\alpha(\bar{m}) = \bar{m}$; thus by Lemma 2.10(4), the map $\beta = \text{id}_{\mathbf{A}} \cup \alpha: \mathbf{C} \rightarrow \mathbf{B}$ is a homomorphism.

$$(2) \quad \gamma: \mathbf{C} \xrightarrow{\text{incl}} \langle \mathbf{Z}^2; \Theta_0, \dots, \Theta_p \rangle \rightarrow \mathbf{V} \cong \lambda \mathbf{V},$$

$$(3) \quad \delta: \lambda \mathbf{V} \times \mathbf{B} \rightarrow \lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B} \cong (\lambda \mathbf{V} \times_{\bar{f}'} \mathbf{B}) + \bar{m} \xrightarrow{\text{incl}} \mathbf{C},$$

$$(4) \quad h: \mathbf{C} \times \mathbf{C} \xrightarrow{\gamma \times \beta} \lambda \mathbf{V} \times \mathbf{B} \xrightarrow{\delta} \mathbf{C}.$$

I claim that the finite algebra $\langle \mathbf{C}; h \rangle$ is nonabelian. Indeed, γ is surjective (since $V + 2\bar{m} \subseteq B_{\bar{0}} + \bar{m} \subseteq C$), so there exist $a, b \in C$ such that $\gamma(a) = \bar{0}$ while $\gamma(b) = \lambda v \neq \bar{0}$. It is easy to check that

$$h(a, c) = f_{\bar{0}}(c) + \bar{m} = f_{\bar{0}}(d) + \bar{m} = h(a, d)$$

but

$$h(b, c) = f_v(c) + \lambda v + \bar{m} \neq f_v(d) + \lambda v + \bar{m} = h(b, d). \quad \square$$

LEMMA 2.12. *There exists a standard M_{p+1} -model $\mathbf{A}$ and points $c, d \in A$ satisfying the hypotheses of Theorem 2.11.*

PROOF. It suffices to assume $p \geq 3$. Let $\mathbf{D}$ be the $\mathcal{L}_p$ -structure given by

$$D = V \setminus \{\bar{0}\}, \\ \theta_i^{\mathbf{D}} = (\theta_i^{\mathbf{V}} \setminus N^2) \cup \Delta_D, \quad \text{where } N = \bar{0}/\theta_i^{\mathbf{V}}.$$

It is not difficult to show that $\theta_i^{\mathbf{D}} \circ \theta_j^{\mathbf{D}} \circ \theta_i^{\mathbf{D}} \circ \theta_j^{\mathbf{D}} = \nabla_D$ if $i \neq j$, so $\mathbf{D}$ is a standard M_{p+1} -model. For each $i = 0, \dots, p-1$ let u_i, v_i be the unique elements of $\{0, \dots, p-1\}$ such that $\langle (1, u_i), \bar{0} \rangle, \langle (2, v_i), \bar{0} \rangle \in \theta_i^{\mathbf{V}}$. Let $c = (0, p)$ and $d = (p, 0)$. Now let $\mathbf{A}$ be the $\mathcal{L}_p$ -structure given by

$$A = D \cup \{c, d\}, \\ \theta_i^{\mathbf{A}} = \text{the reflexive symmetric closure of} \\ \theta_i^{\mathbf{D}} \cup \{ \langle c, (1, u_i) \rangle, \langle d, (2, v_i) \rangle \}, \quad \text{for } i = 0, \dots, p-1, \\ \theta_p^{\mathbf{A}} = \theta_p^{\mathbf{D}} \cup \{ \langle c, c \rangle, \langle d, d \rangle \}.$$

Then $\mathbf{A}$ is a standard M_{p+1} -model which satisfies the hypotheses of Theorem 2.11. $\square$

COROLLARY 2.13. *For every $n \geq 1$ there is a finite nonabelian algebra whose congruence lattice contains a 0,1-copy of M_n . $\square$*

Corollary 2.13 prompts the following question: Does there exist *any* finite lattice L which forces finite algebras $\mathbf{A}$ to be abelian whenever $\text{Con } \mathbf{A}$ has a 0,1-copy of L ? By the embedding theorem of P. Pudlák and J. Tůma [3], there exists a lattice which satisfies this condition if and only if some finite partition lattice Π_n satisfies the condition.

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