EXISTENCE OF AD-NILPOTENT ELEMENTS AND SIMPLE LIE ALGEBRAS WITH SUBALGEBRAS OF CODIMENSION ONE

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ABSTRACT. For a perfect field F of arbitrary characteristic, the following statements are proved to be equivalent: (1) Any Lie algebra over F contains an ad-nilpotent element. (2) There are no simple Lie algebras over F having only abelian subalgebras. They are used to guarantee the existence of an ad-nilpotent element in every Lie algebra over a perfect field of type (C_1) of arbitrary characteristic (in particular, over any finite field). Furthermore, we give a sufficient condition to insure the existence of ad-nilpotent elements in a Lie algebra over any perfect field. As a consequence of this result we obtain an easy proof of the fact that the Zassenhaus algebras and $\mathfrak{sl}(2, F)$ are the only simple Lie algebras which have subalgebras of codimension 1, whenever the ground field F is perfect with $\text{char}(F) \neq 2$. All Lie algebras considered are finite dimensional.

Introduction. It is well known that every Lie algebra over an algebraically closed field contains ad-nilpotent elements. In characteristic zero, it follows from the classical theory. Benkart and Isaacs gave in [3] a proof which works for algebraically closed fields of arbitrary characteristic.

In this paper, we prove that for a nonzero nilpotent derivation D of a Lie algebra L , each element of $\text{Ker } D$ which acts nilpotently on $\text{Ker } D$ is ad-nilpotent on L . We use this result to prove that for a perfect field F the following statements are equivalent: (1) Any Lie algebra over F contains an ad-nilpotent element. (2) There are no simple Lie algebras over F having only abelian subalgebras. Then we show that the following holds: (1) Any Lie algebra over a perfect field of type (C_1) of arbitrary characteristic necessarily contains an ad-nilpotent element. (2) The Brauer group of a perfect field over which every Lie algebra contains ad-nilpotent elements must be trivial (it follows from [6]).

Obviously, Lie algebras containing no nonzero element x such that $\text{ad } x$ is split cannot contain any ad-nilpotent element. The simplest example of these algebras is the 3-dimensional Lie algebra $\mathfrak{su}(2)$ of the real vectors with the vector product.

We prove that the existence of an element x in each subalgebra S of L such that $\text{ad } x$ is split on S guarantees the existence of an ad-nilpotent element in L , provided F is perfect of arbitrary characteristic. Theorem 1.5 below is slightly stronger than this assertion which is obtained from it by taking the subspace V in the statement of the theorem to be zero.

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In §2, we use Theorem 1.5 to study Lie algebras L with a subalgebra of codimension 1 which contains no nonzero ideal of L . These algebras have been studied by several authors, among them Hofmann [8] who showed that such a simple Lie algebra L over the real number field $\mathbb{R}$ must be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Amayo in [1] asserted that over any field F of characteristic $p > 2$, L is either 2-dimensional, isomorphic to $\mathfrak{sl}(2, F)$ or a Zassenhaus algebra. However, Benkart, Isaacs and Osborn gave in [4] counterexamples which show that Amayo's result is not true when F is not perfect. Their results on self-centralizing ad-nilpotent elements yield easy proofs of two special cases of Amayo's assertion: when F is algebraically closed with $\text{char}(F) \neq 2$ and when L contains a self-centralizing ad-nilpotent element provided $\text{char}(F) \neq 2$ (see [4, p. 284]).

By using our Theorem 1.5, we quickly prove that a Lie algebra L as in the preceding paragraph necessarily contains a self-centralizing ad-nilpotent element, whenever F is perfect of arbitrary characteristic. Then, it follows that Amayo's assertion is true when F is perfect with $\text{char}(F) \neq 2$. For simple Lie algebras, this result was also proved in [5] with techniques quite different.

1. On the existence of ad-nilpotent elements. Let L be a finite dimensional Lie algebra over an arbitrary field F . For $x \in L - (0)$, we will denote by $E_L(x)$ the Engel subalgebra of L relative to x ; that is, $E_L(x)$ is the Fitting null-component of L relative to $\text{ad } x$.

First we consider the kernel of a nilpotent derivation of L .

LEMMA 1.1. *Let D be a nonzero nilpotent derivation of a Lie algebra L over an arbitrary field. Assume that $x \in \text{Ker } D$ acts nilpotently on $\text{Ker } D$. Then x acts nilpotently on L .*

PROOF. Suppose that $\text{ad } x$ is not nilpotent on L . Let $L = E_L(x) \oplus L_1(x)$ be the Fitting decomposition of L relative to $\text{ad } x$. We have $L_1(x) \neq 0$. Since $x \in \text{Ker } D$ we find

$$[\text{ad } x, D] = \text{ad } D(x) = 0$$

so that D and $\text{ad } x$ commute. Then $L_1(x)$ is stabilized by D . Since D is nilpotent, we have that $\text{Ker } D \cap L_1(x) \neq 0$. On the other hand, since x acts nilpotently on $\text{Ker } D$ it follows that $\text{Ker } D \leq E_L(x)$. We find, $0 \neq \text{Ker } D \cap L_1(x) \leq E_L(x) \cap L_1(x) = 0$, a contradiction.

THEOREM 1.2. *Let F be an arbitrary perfect field. Then the following statements are equivalent:*

- (1) *Any Lie algebra over F contains ad-nilpotent elements.*
- (2) *There are no simple Lie algebras over F having only abelian subalgebras.*

PROOF. To prove (1) implies (2). Suppose that L is a simple Lie algebra over F having only abelian subalgebras. By using Theorem 4.1 of [6] we obtain that $\text{ad } x$ is semisimple for every $x \in L$. By (1), there exists an element $y \in L - (0)$ such that $\text{ad } y$ is nilpotent on L . This yields $y \in Z(L) = 0$, a contradiction.

In order to prove the converse, suppose that L is a Lie algebra over F which contains no nonzero ad-nilpotent elements and L is of minimal dimension. First we claim that $\text{ad } x$ is semisimple for every $x \in L$. To prove this, let D be a nonzero nilpotent derivation of L . Then $\text{Ker } D$ is a proper subalgebra of L . By the minimality of $\dim L$, we have that $\text{Ker } D$ contains a nonzero element z such that

$\text{ad } z$ is nilpotent in $\text{Ker } D$. From Lemma 1.1 it follows that $\text{ad } z$ is nilpotent on L , a contradiction. Therefore, L has no nonzero nilpotent derivation. The claim now follows from the Jordan-Chevalley decomposition (see [7, Chapter 2, Theorem 3.5 and Proposition 3.3]).

Now let S be a proper subalgebra of L . Assume $Z(S) \neq S$. Then as $\dim S/Z(S) < \dim L$, we have that there exists $u \in S - Z(S)$ such that u acts nilpotently on $S/Z(S)$. We see that u acts nilpotently on S . Since $\text{ad } x$ is semisimple by the claim, we conclude that $u \in Z(S)$ which is a contradiction. Therefore, S is abelian.

Finally, we show that L is simple. Suppose that N is a proper ideal of L . Let $x \in N - (0)$. Then $[x, L] \leq N$ whence $[x[x, L]] = 0$ since N is abelian. This yields $\text{ad } x$ is nilpotent on L , a contradiction. This completes the proof.

A field F is said to be a *field of type (C_1)* if every homogeneous polynomial $f(X_1, \dots, X_n)$ over F of degree less than the number n of variables has a nontrivial root in F^n .

We recall that every finite field is of type (C_1) by Chevalley's Theorem (see [10]).

COROLLARY 1.3. *Let F be a perfect field of type (C_1) of arbitrary characteristic. Then every Lie algebra over F contains ad-nilpotent elements.*

PROOF. Suppose that the assertion is not true. Then, by Theorem 1.2, there exists a simple Lie algebra L over F having only abelian subalgebras. Let $x \in L - (0)$. By [6], $\text{ad } x$ is semisimple and so $E_L(x)$ is a proper subalgebra of L . Since $E_L(x)$ is self-normalizing by [2], it follows that $E_L(x)$ is a Cartan subalgebra of L .

Now suppose $\text{char}(F) = 0$. Then, from the classical theory it follows that all Cartan subalgebras of L have the same dimension. On the other hand, we consider the characteristic polynomial $f(\lambda)$ of L relative to a basis (e_i) of L (see [9]). Write

$$(*) \quad f(\lambda) = \lambda^n - \tau_1(X_1, \dots, X_n)\lambda^{n-1} + \dots + (-1)^l \tau_{n-l}(X_1, \dots, X_n)\lambda^l$$

where $\tau_i(X_1, \dots, X_n)$ is a homogeneous polynomial of degree i . Take an element $x \in L - (0)$ and decompose $x = t_1 e_1 + \dots + t_n e_n$ with $t_i \in F$. Then, the characteristic polynomial of $\text{ad } x$ is obtained by specializing $X_i = t_i$, $i = 1, \dots, n$ in $(*)$. Pick r such that $\tau_{n-r}(t_1, \dots, t_n) \neq 0$ and $\tau_{n-i}(t_1, \dots, t_n) = 0$ for $i > r$. We have $\dim E_L(x) = r$. Now consider the polynomial $\tau_{n-r}(X_1, \dots, X_n)$. As F is of type (C_1) , we can take a nontrivial root $(s_1, \dots, s_n) \in F^n$ of τ_{n-r} . Write $y = s_1 e_1 + \dots + s_n e_n$. We find $\dim E_L(y) \neq r$ so $\dim E_L(x) \neq \dim E_L(y)$, which is a contradiction.

Suppose then $\text{char}(F) = p > 0$. Now we consider the p -closure C of L in $\text{Der}(L)$, where $\text{Der}(L)$ denotes the derivation algebra of L . The proof of Theorem 7.2 of [6] shows that C is p -simple and every proper p -subalgebra of C is abelian. Assume that u is a nonzero nilpotent element of C . Since L is an ideal of C , we have that u acts nilpotently on L . As L has no nonzero nilpotent derivation by Theorem 4.1 of [6], we find $[uL] = 0$. But clearly the centralizer $C_C(L)$ of L in C is a p -ideal of L , so $C_C(L) = 0$ whence $u = 0$, which is a contradiction. We conclude that C contains no nonzero nilpotent elements. Then, since F is perfect, we have that every element of the restricted Lie algebra C is semisimple.

Now, let $x \in C - (0)$ and consider the Engel subalgebra $E_C(x)$. We have that $E_C(x)$ is a p -subalgebra since it is self-normalizing. Then, by above, $E_C(x)$ is a toral Cartan subalgebra of C . From [11] it follows that all Engel subalgebras of C have the same dimension. On the other hand, as in the characteristic zero case, we can

find two elements $x, y \in C$ such that $\dim E_C(x) \neq \dim E_C(y)$. This contradiction completes the proof.

COROLLARY 1.4. *Let F be a perfect field over which every Lie algebra contains an ad-nilpotent element. Then the Brauer group $\text{Br}(F)$ of F is trivial.*

PROOF. This result follows from Theorem 1.2 and Theorem 8.5 of [6].

Next, by using Lemma 1.1, we obtain a sufficient condition to insure the existence of an ad-nilpotent element in a suitable subset of a Lie algebra over a perfect field of arbitrary characteristic.

THEOREM 1.5. *Let L be a Lie algebra over a perfect field F of arbitrary characteristic. Assume that every subalgebra T of L contains at least one element $x \neq 0$ with $\text{ad } x$ split on T . Let V be a vector subspace of L consisting of ad-split elements. Assume for every $y \in L - V$ that V contains no nonzero subspace invariant under $\text{ad } y$. Then every subalgebra S of L not contained in V has a nonzero element $z \in S - V$ such that $\text{ad } z$ is nilpotent on S .*

PROOF. Let S be a subalgebra of L such that $S \not\subseteq V$. We argue by induction on $\dim S$. For $\dim S = 1$ it is trivial. Then suppose $\dim S = r > 1$ and the result holds for subalgebras of dimension less than r . Let $x \in S - V$ and let $\text{ad}_S x = D_n + D_s$ be the Jordan-Chevalley decomposition of $\text{ad}_S x$. Assume $D_n \neq 0$. Then $\text{Ker } D_n$ is a proper subalgebra of S . We have,

$$\text{ad}_S(D_n(x)) = [\text{ad}_S x, D_n] = [D_s, D_n] = 0$$

whence $D_n(x) \in Z(S)$. But $Z(S) \cap V = 0$ by our hypothesis on V . This yields $D_n(x) = 0$, otherwise we would have $D_n(x) \notin V$ with $D_n(x)$ acting nilpotently on S . Therefore, $x \in \text{Ker } D_n$ so that $\text{Ker } D_n \not\subseteq V$. Now, by the inductive hypothesis, there exists $y \in \text{Ker } D_n - V$ such that $\text{ad } y$ is nilpotent on $\text{Ker } D_n$. But then from Lemma 1.1 it follows that $\text{ad } y$ is nilpotent on S . We may assume then that $\text{ad } x$ is semisimple for every $x \in S - V$.

Next, take $z \in S$ such that $\text{ad } z$ is split on S . Assume $z \notin V$. Let α be an eigenvalue of $\text{ad } z$. We have $[xz] = \alpha x$ for some $x \in S - 0$. We find $[x[xz]] = 0$ so that $(\text{ad}_S x)^2(z) = 0$. But since $z \notin V$ it follows that $x \notin V$ by our hypothesis on V again. Then, by above, $\text{ad}_S x$ is semisimple and hence $(\text{ad } x)(z) = 0$. This yields $\alpha = 0$, thus $\text{ad } z$ is nilpotent on S . We may suppose then that $z \in V$. In particular we have that $S \cap V \neq 0$ and then $Z(S) = 0$ by our hypothesis on V .

Now let T be a subalgebra of L maximal with respect to be contained in $S \cap V$. By above, $T \neq 0$. If every element of T acts nilpotently on S , then from Engel's Theorem it follows that there exists $u \in S - T$ such that $[u, T] \leq T$. But then, since T is invariant under $\text{ad } u$, we find that $u \in V$ by our hypothesis on V . This yields that the subalgebra $T + Fu$ is contained in $S \cap V$, which contradicts the maximality of T . We conclude that there exists an element $v \in V \cap S$ such that $\text{ad } v$ is not nilpotent on S .

Finally, let D be the semisimple component of $\text{ad}_S v$. As $v \in V$, we have that $\text{ad}_S v$ is split. Thus D is split too, and $D \neq 0$ since $\text{ad}_S v$ is not nilpotent. Let $S = S_0(D) \oplus S_\alpha(D) \oplus \cdots \oplus S_\gamma(D)$ be the decomposition of S into the eigenspace of D . We have $S_0(D) = \text{Ker } D$. Suppose $\text{Ker } D \not\subseteq V$. Then, since $\text{Ker } D$ is a proper subalgebra of S there exists $x \in (\text{Ker } D) - V$ such that $\text{ad } x$ is nilpotent on $\text{Ker } D$

by the inductive hypothesis. Since $x \notin V$, we have that $\text{ad } x$ is semisimple on S . It follows that x acts trivially on $\text{Ker } D$. Thus $(\text{Ker } D) \cap V$ is invariant under $\text{ad } x$, so $(\text{Ker } D) \cap V = 0$ by our hypothesis on V . But, $\text{ad}_S(D(v)) = [\text{ad}_S v, D] = 0$, whence $D(v) \in Z(S) = 0$, so that $v \in (\text{Ker } D) \cap V$, which is a contradiction. We conclude that $S_0(D) \leq V$. This yields $S_\alpha(D) \not\leq V$ for some $\alpha \neq 0$ since $S \not\leq V$ and V is a vector subspace of L . Pick $e \in S_\alpha(D) - V$. We have $D(e) = \alpha e$ and then $[\text{ad}_S e, D] = \text{ad}_S(D(e)) = \alpha(\text{ad}_S e)$, whence $[\text{ad}_S e[\text{ad}_S e, D]] = 0$. On the other hand, since $e \in S - V$ we find that $\text{ad}_S e$ is semisimple, and thus $\text{ad}(\text{ad}_S e): \text{Der}(S) \rightarrow \text{Der}(S)$ is semisimple. This yields $[\text{ad}_S e, D] = \text{ad}_S(D(e)) = 0$ so $D(e) = 0$ and hence $\alpha = 0$. This contradiction completes the proof.

COROLLARY 1.6. *Let L be a Lie algebra over a perfect field of arbitrary characteristic. Assume that every subalgebra S of L contains at least one element x such that $\text{ad } x$ is split on S . Then every subalgebra T of L contains an element y with $\text{ad } y$ nilpotent on T .*

PROOF. It follows from Theorem 1.5 by taking V to be zero.

2. Lie algebras with subalgebras of codimension one. In this section we consider a situation in which Theorem 1.5 applies. By using this Theorem, we quickly prove that a Lie algebra L with a subalgebra M of codimension 1 which contains no nonzero ideals of L necessarily contains a self-centralizing ad-nilpotent element, whenever the ground field F is perfect of arbitrary characteristic. From this result and Theorem 3.2 of [4] follows an easy proof of Amayo's assertion, whenever F is perfect with $\text{char}(F) \neq 2$.

The proof of the following Lemma is reproduced from the proof of Lemma 2.1 of [1]. We include it here for completeness.

LEMMA 2.1. *Let L be a finite dimensional Lie algebra over an arbitrary field F . Suppose $M \leq L$ is a subalgebra of codimension one which contains no nonzero ideals of L . Then M is supersolvable.*

PROOF. Define $M_0 = M$, $M_{i+1} = \{x \in M_i \mid [xL] \leq M_i\}$ for $i \geq 0$. We have $[M_i, M_j] \leq M_{i+j}$, in particular $M_i \leq M$ for all i . If $M_i = M_{i+1}$, then M_{i+1} is an ideal of L , thus M_{i+1} must be zero and hence $M_i = 0$. Then we have $M = M_0 > M_1 > \dots > M_r = 0$ for some r .

Now let $a \in L - M$. We claim that $M_{i+1} = \{x \in M_i \mid (\text{ad } a)^{i+1}(x) \in M\}$ for $i \geq 0$. It is clear for $i = 0$ since $L = M + Fa$. Suppose that $i \geq 1$ and the result holds for i . If $x \in M_i$ with $(\text{ad } a)^{i+1}(x) \in M$, then $(\text{ad } a)^i[xa] \in M$ and $[xa] \in M_{i-1}$, whence $[xa] \in M_i$. We have $[xL] \leq [xM] + [x, Fa] \leq [M_i, M] + M_i = M_i$, so that $x \in M_{i+1}$. The claim follows.

Next, for $i \geq 0$ we define the linear map $\sigma_i: M \rightarrow F$ by means of $(\text{ad } a)^{i+1}(x) \equiv \sigma_i(x)a \pmod{M}$. We have $\text{Ker } \sigma = M_{i+1}$ by above. Therefore, $\dim M_i/M_{i+1} = 1$ for $i = 0, 1, \dots, r-1$. Thus M is supersolvable.

THEOREM 2.2. *Let L be a Lie algebra over a perfect field F of arbitrary characteristic. Suppose $M \leq L$ is a subalgebra of codimension one which contains no nonzero ideals of L . Then L contains a self-centralizing ad-nilpotent element.*

PROOF. By Lemma 2.1, M is supersolvable so that $\text{ad } y$ is split on M for every $y \in M$. As $\dim L/M = 1$, we have that $\text{ad } y$ is split on L for every $y \in M$. On

the other hand, the proof of Lemma 3.7 of [4] shows for every $x \in L - M$ that M contains no nonzero subspace invariant under $\text{ad } x$.

Now, let S be a subalgebra of L such that $S \not\subseteq M$. If $\dim S = 1$, write $S = Fx$, then obviously, $\text{ad } x$ is split on S . Now suppose $\dim S > 1$. Since $\dim L/M = 1$, we have $S \cap M \neq 0$. Then S contains a nonzero element y such that $\text{ad } y$ is split on S .

Therefore, Theorem 1.5 applies and there exists $x \in L - M$ such that $\text{ad } x$ is nilpotent.

Finally, by Lemma 3.8 of [4] we have that $C_L(x) = Fx$. The proof is complete.

COROLLARY 2.3. *Let L be a Lie algebra over a perfect field F with $\text{char}(F) \neq 2$. Assume that L has a subalgebra of codimension 1 which contains no nonzero ideals of L . Then, L is either 2-dimensional, isomorphic to $\text{sl}(2, F)$ or a Zassenhaus algebra.*

PROOF. It follows from Theorem 2.2 and Theorem 3.2 of [4].

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