POLYNOMIALS AND NUMERICAL RANGES

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Let A be an $n \times n$ complex matrix. For $1 \leq k \leq n$ we study the inclusion relation for the following polynomial sets related to the matrix A .

- (a) The classical numerical range of the k th compound of the matrix $\lambda I - A$.
- (b) The k th decomposable numerical range of the matrix $\lambda I - A$.
- (c) The convex hull of the set of all monic polynomials of degree k that divide the characteristic polynomial of A .

Moreover, we give an example showing that the set described in (a) is not convex in general. This settles a question raised by C. Johnson.

Results and examples. Let $C_{n \times n}$ be the set of all $n \times n$ complex matrices. For $A \in C_{n \times n}$, its (*classical*) *numerical range* is the set

$$W(A) = \{xAx^* : x \in C^n \text{ and } xx^* = 1\}.$$

Suppose A is hermitian and its characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$ has roots $\lambda_1 \geq \dots \geq \lambda_n$. For $1 \leq k \leq n$, we denote by $\text{Int}(p_A(\lambda); k)$ the set of all monic polynomials $q(\lambda)$ of degree k that have k real roots $\mu_1 \geq \dots \geq \mu_k$ satisfying

$$\lambda_i \geq \mu_i \quad \text{and} \quad \mu_{k-i+1} \geq \lambda_{n-i+1} \quad \text{for } i = 1, \dots, k.$$

Let $P_k(\lambda I - A)$ be the convex hull of the set

$$\left\{ \prod_{j=1}^k (\lambda - \lambda_{i_j}) : 1 \leq i_1 < \dots < i_k \leq n \right\}.$$

Johnson [3] proved the interesting result that

$$(1) \quad \text{Int}(p_A(\lambda); n-1) = W(\text{adj}(\lambda I - A)) = P_{n-1}(\lambda I - A).$$

For $B \in C_{n \times n}$ and $1 \leq k \leq n$, let $C_k(B)$ be the k th compound of B (see [4] for definitions and properties). Since $W(B^t) = W(B)$ and $C_{n-1}(B)$ is unitarily similar to $\text{adj}(B)^t$, we may replace " $W(\text{adj}(\lambda I - A))$ " by " $W(C_{n-1}(\lambda I - A))$ " in (1). Clearly,

$$\begin{aligned} \text{Int}(p_A(\lambda); 1) &= \{\lambda - \mu : \lambda_1 \geq \mu \geq \lambda_n\} \\ &= \{\lambda - \mu : \mu \in W(A)\} \\ &= W(\lambda I - A) = W(C_1(\lambda I - A)), \end{aligned}$$

Received by the editors August 10, 1987 and, in revised form, November 20, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 15A60, 15A69, 26C10.

Key words and phrases. Numerical range, decomposable numerical range, characteristic polynomial.

Research supported by NSF grant DMS-8521521.

and

$$\text{Int}(p_A(\lambda); n) = \{p_A(\lambda)\} = W(C_n(\lambda I - A)).$$

We have

$$(2) \quad \text{Int}(p_A(\lambda); k) = W(C_k(\lambda I - A)) = P_k(\lambda I - A)$$

for $k = 1, n - 1, n$.

Related to $W(C_k(B))$ is the concept of the k th decomposable numerical range of the matrix B defined by (see [5])

$$W_k^\wedge(B) = \left\{ x C_k(B) x^* : x \in \bigwedge^k \mathbb{C}^n \text{ is decomposable and } x x^* = 1 \right\},$$

where $\bigwedge^k \mathbb{C}^n$ is the k th Grassman space over $\mathbb{C}^n$. Evidently,

$$(3) \quad W_k^\wedge(B) \subseteq W(C_k(B)).$$

It is known (see [4]) that equality in (3) holds for $k = 1, n - 1, n$. So (2) may be rewritten as

$$(4) \quad \text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A) = W(C_k(\lambda I - A)) = P_k(\lambda I - A)$$

for $k = 1, n - 1, n$.

For general k we have the following theorem.

THEOREM 1. *Let $A \in \mathbb{C}_{n \times n}$ be hermitian. Then for $1 \leq k \leq n$,*

$$\text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A) \subseteq W(C_k(\lambda I - A)) = P_k(\lambda I - A).$$

All four sets are equal if and only if the set

$$\text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A)$$

is convex.

PROOF. For $B \in \mathbb{C}_{n \times n}$, let $B[k]$ denote the $k \times k$ leading principal submatrix of B . Then (see [5]),

$$W_k^\wedge(\lambda I - A) = \{\det((\lambda I - UAU^*)[k]) : U \text{ unitary}\}.$$

It follows that $q(\lambda) \in W_k^\wedge(\lambda I - A)$ if and only if $q(\lambda)$ is the characteristic polynomial of a matrix of the form $(UAU^*)[k]$, where U is a unitary matrix. By a result of Fan and Pall [2],

$$\text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A).$$

The inclusion

$$W_k^\wedge(\lambda I - A) \subseteq W(C_k(\lambda I - A))$$

is clear. Now consider $W(C_k(\lambda I - A))$. Since

$$W(C_k(\lambda I - A)) = W(C_k(U(\lambda I - A)U^*))$$

for any unitary matrix U , we may assume that

$$\lambda I - A = \text{diag}(\lambda - \lambda_1, \dots, \lambda - \lambda_n).$$

It follows that $C_k(\lambda I - A)$ is a diagonal matrix with diagonal entries $\prod_{j=1}^k (\lambda - \lambda_{i_j})$ with $1 \leq i_1 < \dots < i_k \leq n$. One easily checks that the numerical range of a diagonal matrix equals the convex hull of the diagonal entries. So

$$W(C_k(\lambda I - A)) = P_k(\lambda I - A).$$

Finally, if the four sets are equal, then clearly the set $\text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A)$ is convex. Conversely, suppose $\text{Int}(p_A(\lambda); k) = W_k^\wedge(\lambda I - A)$ is convex. By the fact that

$$\prod_{j=1}^k (\lambda - \lambda_{i_j}) \in \text{Int}(p_A(\lambda); k) \quad \text{for } 1 \leq i_1 < \cdots < i_k \leq n,$$

we have

$$P_k(\lambda I - A) \subseteq \text{Int}(p_A(\lambda); k),$$

and hence the four sets are equal. $\square$

We show that the four sets are not equal in general by the following example.

EXAMPLE 1. Let $A = \text{diag}(1, 2, 3, 4)$. Then

$$p(\lambda) = (\lambda - 1)(\lambda - 2), \quad q(\lambda) = (\lambda - 3)(\lambda - 4) \in \text{Int}(p_A(\lambda); 2).$$

However, the polynomial $(p(\lambda) + q(\lambda))/2$ has no real roots and thus does not belong to $\text{Int}(p_A(\lambda); 2)$. So

$$\text{Int}(p_A(\lambda); 2) \neq P_2(\lambda I - A).$$

As mentioned in the proof of Theorem 1, $W_k^\wedge(\lambda I - A)$ is the set of all characteristic polynomials of matrices in the form $(UAU^*)[k]$, where U is unitary. For a hermitian matrix A , $W_k^\wedge(\lambda I - A)$ is simply $\text{Int}(p_A(\lambda); k)$. For general matrices A , we do not have such a nice equivalence form for $W_k^\wedge(\lambda I - A)$. Anyway, one may still consider the sets $W(C_k(\lambda I - A))$ and $P_k(\lambda I - A)$. In particular, we may extend Theorem 1 to

THEOREM 2. Let $A \in \mathbb{C}_{n \times n}$ be a normal matrix. Then

$$W_k^\wedge(\lambda I - A) \subseteq W(C_k(\lambda I - A)) = P_k(\lambda I - A).$$

The three sets are equal if and only if the set $W_k^\wedge(\lambda I - A)$ is convex.

PROOF. Similar to that of Theorem 1. $\square$

For nonnormal matrices A , we have no inclusions between the sets $W(C_k(\lambda I - A))$ and $P_k(\lambda I - A)$. Moreover, the set $W(C_k(\lambda I - A))$ may fail to be convex as shown in the following examples.

EXAMPLE 2. Let

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then

$$P_1(\lambda I - A) = \{\lambda\} \subsetneq \{\lambda - \mu: |\mu| \leq 1\} = W(C_1(\lambda I - A)).$$

EXAMPLE 3. Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\lambda I - A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & -1 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

and

$$C_3(\lambda I - A) = \lambda^3 I + \lambda^2 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that

$$p(\lambda) = \lambda^3 + a\lambda^2 + \frac{1}{2}\lambda \in W(C_3(\lambda I - A))$$

if and only if there exist $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ such that

$$\begin{aligned} xx^* &= 1; \\ x_1 \bar{x}_4 &= 0; \\ x_1 \bar{x}_3 + x_1 \bar{x}_4 + x_2 \bar{x}_4 &= \frac{1}{2}; \\ x_1 \bar{x}_2 + x_1 \bar{x}_3 + x_2 \bar{x}_3 + x_3 \bar{x}_4 &= a. \end{aligned}$$

By the second equation, $x_1 = 0$ or $x_4 = 0$. Substituting back to the first and third equations, we have either

$$(i) \ x_1 \bar{x}_3 = \frac{1}{2} \text{ and } x_2 = x_4 = 0; \text{ or}$$

$$(ii) \ x_2 \bar{x}_4 = \frac{1}{2} \text{ and } x_1 = x_3 = 0.$$

It follows from the fourth equation that either $a = \frac{1}{2}$ or $a = 0$. So

$$p_1(\lambda) = \lambda^3 + \frac{1}{2}\lambda^2 + \frac{1}{2}\lambda, \quad p_2(\lambda) = \lambda^3 + \frac{1}{2}\lambda \in W(C_3(\lambda I - A))$$

whereas $(p_1(\lambda) + p_2(\lambda))/2 \notin W(C_3(\lambda I - A))$. Hence $W(C_3(\lambda I - A))$ is not convex and cannot be equal to $P_3(\lambda I - A)$.

Concluding remarks. 1. Example 3 gives a negative answer to the question about the convexity of the set $W(\text{adj}(\lambda I - A))$ raised by Johnson [3].

2. Observe that $C_k(\lambda I - A)$ is actually a matrix polynomial. Instead of looking at $W(C_k(\lambda I - A))$, one may consider the numerical range of a matrix polynomial

$$p(A) = \lambda^m A_0 + \cdots + \lambda A_{m-1} + A_m.$$

Clearly, the set

$$W(p(A)) = \{x(p(A))x^*: x \in \mathbb{C}^n \text{ and } xx^* = 1\}$$

is essentially the *joint range* of the matrices $A_0, \dots, A_m$, defined by

$$\mathscr{W}(A_0, \dots, A_m) = \{(xA_0x^*, \dots, xA_mx^*): x \in \mathbb{C}^n \text{ and } xx^* = 1\}.$$

As indicated in [1], the set $\mathscr{W}(A_0, \dots, A_m)$ may fail to be convex even for hermitian matrices $A_0, \dots, A_m$, if $m > 2$.

3. Note that $C_k(A)$ is a particular type of induced matrix (see [4]). Our results may be generalized to other classes of induced matrices.

ACKNOWLEDGMENT. Thanks are due to Nam-Kiu Tsing for his helpful discussion.

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