MODULAR GROUP ALGEBRAS OF SIMPLY PRESENTED ABELIAN GROUPS

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ABSTRACT. Let G be a p -local Warfield group and F a perfect field of characteristic p . It is shown that if the group algebras FG and FH are isomorphic, then G and H are isomorphic.

1. Introduction. Let p be a prime number. Throughout this paper F will denote a perfect field of characteristic p , and G will denote an abelian group which will be written multiplicatively.

If H is a group such that the group algebras FG and FH are F -isomorphic, then it has been known for some time that the Ulm p -invariants of G and H must be equal [2, 5, 9]. Consequently, G and H are isomorphic if they are simply presented abelian p -groups. We remark that the Ulm invariant at p -height ∞ is included above, thus G and H need not be reduced. Suppose, however, one hypothesizes that only G is a simply presented p -group. The question then arises whether H must be one also. This question has resisted resolution except in special cases. If G is countable, then H is necessarily simply presented. If G is a direct sum of cyclic p -groups, then Berman and Mollov [3] proved that H is one also. It was then shown in [10] that if G is simply presented of p -length less than the first uncountable ordinal, then the same is true of H . This restriction on length was due to the use of a theorem of P. Hill in which such a limitation was necessary.

In this paper, we shall settle the question for simply presented p -groups by using a more direct approach. We let $V(G)$ denote the group of normalized units (i.e. of augmentation 1) in FG . Our results will be consequences of

THEOREM 1. *$V(G)$ is a simply presented abelian p -group if and only if G is a simply presented abelian p -group.*

The Ulm invariants of $V(G)$ have been computed for F a perfect field, therefore the structure of $V(G)$ is determined for G simply presented.

Before stating an isomorphism theorem, we give an immediate application of Theorem 1 to the direct factor problem. This problem asks whether G is a direct factor of $V(G)$. Discussions of this question can be found in [4 and 11]. For simply presented p -groups, the result on splitting proved in [1, Theorem 5.2] is subsumed by

THEOREM (DIRECT FACTOR). *If G is a simply presented abelian p -group, then G is a direct factor of $V(G)$.*

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We now give an isomorphism theorem in a more general setting than simply presented p -groups. The p -local Warfield groups are groups that may be mixed, and which are characterized as summands of simply presented modules over the ring of p -adic rational numbers. A reference for p -local Warfield groups is [7].

THEOREM 3 (ISOMORPHISM). *If G is a p -local Warfield group and H is a group such that $FG \cong FH$, then $G \cong H$.*

COROLLARY. *If G is a simply presented abelian p -group and H is a group such that $FG \cong FH$, then $G \cong H$.*

We remark that it was shown in [1] that this theorem holds if H is also assumed to be a p -local Warfield group.

Two questions which these results raise are the following. Does the direct factor theorem hold for p -local Warfield groups G ? This could easily be proved provided $V(G)$ is a Warfield group if G is one; however, this is probably not true. To state the second question, let tG denote the torsion subgroup of an abelian group G . If tG is a simply presented p -group, and if $FG \cong FH$, then is $tG \cong tH$?

Finally, we wish to point out the importance of the isomorphism problem for groups with only p -torsion over characteristic p fields. It is shown by Ullery in [14] that resolution of this problem is central to that for abelian group algebras over arbitrary coefficient rings. Unfortunately, the likelihood of settling these matters in full generality seems quite remote.

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2. Characteristic p lemmas. As before, F will denote a perfect field of characteristic p . If $N \leq G$, then the natural map $FG \rightarrow F(G/N)$ of group algebras induces a homomorphism $V(G) \rightarrow V(G/N)$ whose kernel we denote by $K(N)$. Letting $I(N)$ denote the augmentation ideal of FN , we can write $K(N) = V(G) \cap (1 + FG \cdot I(N))$.

LEMMA 1. *If tG denotes the torsion subgroup of G , then $V(G) = G \cdot K(tG)$ and $G \cap K(tG) = tG$.*

PROOF. Since G/tG is torsion-free, it is well known that $V(G/tG) = G/tG$, hence the image of G under the natural map $V(G) \rightarrow V(G/tG)$ is the whole image. Thus $V(G) = G \cdot K(tG)$. The second statement is obvious.

LEMMA 2. (1) *Let $N \leq G$, N a p -group. Under the natural map $FG \rightarrow F(G/N)$, any preimage of an element of $V(G/N)$ lies in $V(G)$. The map $V(G) \rightarrow V(G/N)$ is surjective, and $K(N) = 1 + FG \cdot I(N)$ is a p -group.*

(2) *If G is a p -group, then $V(G) = 1 + I(G)$ is a p -group.*

(3) *If tG is a p -group, then $V(G)/G$ is a p -group.*

PROOF. The kernel of $FG \rightarrow F(G/N)$ is $F(G) \cdot I(N)$, which is a nil ideal since $(\sum a_i g_i)^{p^n} = 0$ if $a_i \in F$, $g_i \in N$, $\sum a_i = 0$ and $g_i^{p^n} = 1$ ($1 \leq i \leq k$). Thus any preimage of a unit is a unit. The rest of (1) now follows easily. Clearly, (1) implies (2) since $K(G) = V(G)$. In the situation of (3), (1) implies that $K(tG)$ is a p -group, thus Lemma 1 proves (3).

We remark on a representation for elements of $K(N)$ that we will use later. Under the assumption of (1) above, suppose that $\{g_i : i < \nu\}$ is a family of coset

representatives of N in G for some ordinal ν . Then the elements of $K(N)$ are precisely the sums $1 + \sum_{i < \nu} g_i \alpha_i$, $\alpha_i \in I(N)$ with almost all $\alpha_i = 0$. Moreover, the α_i in this representation are unique.

Next we wish to discuss p -heights for abelian groups and commutative rings of characteristic p . We shall refer to these as heights since p is fixed. If S is either a group or ring as above, define S^σ for every ordinal σ in the usual manner: $S^0 = S$; $S^{\sigma+1} = (S^\sigma)^p$; and $S^\sigma = \bigcap_{\tau < \sigma} S^\tau$ if σ is a limit ordinal. If σ is the first ordinal such that $S^\sigma = S^{\sigma+1}$, we call σ the p -length of S and define S^∞ to be S^σ . Then S^∞ is the maximal p -divisible subgroup or subring of S . If $s \in S$, we define the height $h(s)$ to be ∞ if $s \in S^\infty$, and the smallest ordinal σ such that $s \notin S^{\sigma+1}$ otherwise. Note that $F^\sigma = F$ for every σ since F is perfect. Since $(\sum a_i g_i)^p = \sum a_i^p g_i^p$, it is easily shown that $(FG)^\sigma = FG^\sigma$ and $V(G)^\sigma = V(G) \cap FG^\sigma$. Thus $h(\sum a_i g_i) = \min\{h(g_i)\}$, and the notions of height in G , $V(G)$ and FG agree whenever a comparison can be made.

LEMMA 3. *Let A be a subgroup of G . The following are equivalent:*

- (1) A is isotype in G ;
- (2) A is isotype in $V(G)$;
- (3) $V(A)$ is isotype in $V(G)$.

PROOF. The equivalence of (1) and (2) follow from the remark above on heights in G and $V(G)$. To show that (1) and (3) are equivalent, we first note that if $B \leq G$, then $V(A) \cap FB = V(A \cap B)$ (see [8, p. 37]). Thus $V(A) \cap V(G)^\sigma = V(A) \cap FG^\sigma = V(A \cap G^\sigma)$, and $V(A)^\sigma = V(A) \cap FA^\sigma = V(A^\sigma)$. Equality will hold precisely when $A^\sigma = A \cap G^\sigma$.

LEMMA 4. (1) *Let $N \leq G$, N a p -group. Then N is nice in G if and only if $K(N)$ is nice in $V(G)$.*

(2) *Let $N \leq G$. Then N is nice in G if and only if N is nice in $V(G)$.*

PROOF. (1) By Lemma 2, $V(G)/K(N) = V(G/N)$. Assume that N is nice in G and that $\sum a_i (g_i N) \in V(G/N)$. We may suppose the g_i are chosen such that $h(g_i) = h(g_i N)$. (It should be clear from context in what groups heights are computed.) By Lemma 2, $\sum a_i g_i \in V(G)$ and $h(\sum a_i g_i) = h(\sum a_i (g_i N))$. Thus $K(N)$ is nice in $V(G)$.

For the converse, let $g \in G$. Since $K(N)$ is nice in $V(G)$, we may choose $\alpha \in K(N)$ such that $h(g\alpha) = h(gN)$. By examining the representation of α given after Lemma 2, the support of α must contain an element $n \in N$. But $h(gn) \geq h(g\alpha) = h(gN)$, thus $h(gn) = h(gN)$ and N is nice in G .

(2) By Lemma 3, G is isotype in $V(G)$. Therefore it suffices to show that G is nice in $V(G)$. Let $\alpha \in V(G)$ and consider $G\alpha$. Altering α by a factor from G , we may assume that e lies in the support of α . Suppose that $g \in G$ with $h(g) = h(\alpha)$. Then clearly $h(g\alpha) \leq h(g) = h(\alpha)$, thus G is nice in $V(G)$.

3. Simply presented G . Let $A \leq B \leq G$. By a *smooth chain* from A to B we mean a family of subgroups $\{N_\sigma : \sigma \leq \lambda\}$ for some ordinal λ , such that $N_0 = A$, $N_\lambda = B$, $N_\tau \subseteq N_\sigma$ if $\tau < \sigma$, and $N_\sigma = \bigcup_{\tau < \sigma} N_\tau$ if σ is a limit ordinal. If, in addition, $(N_{\sigma+1} : N_\sigma) = p$ for every $\sigma < \lambda$, we say we have a *composition series* from A to B . We call it a *nice composition series* if every N_σ is nice in G . Then G is simply presented if there exists a nice composition series from 1 to G .

Now let $S \leq G$, S a p -group. We want to define certain subgroups of $K(S)$ in order to construct a nice composition series. Let $M = I(S)$ be the augmentation ideal of FS , and choose a family $\{g_i: i < \nu\}$ of coset representatives of S in G . By the remark after Lemma 2, each $\alpha \in K(S)$ has a unique representation $\alpha = 1 + \sum_{i < \nu} g_i \alpha_i$, where $\alpha_i \in M$. Note that $h(\alpha) = \min\{h(g_i \alpha_i): i < \nu\}$ since the supports of the $g_i \alpha_i$ are disjoint. Let $k \geq 1$, $\sigma \leq \nu$. We define

$$N_{k,\sigma} = 1 + \sum_{i < \sigma} g_i M^k + \sum_{\sigma \leq i < \nu} g_i M^{k+1}$$

where M^k is the ideal product in FS . It is easy to see that $N_{k,\sigma}$ is closed under multiplication since $g_i M^a g_j M^b \subseteq g_m M^{k+1}$ if $a, b \in \{k, k+1\}$ and $g_i g_j S = g_m S$. Since $K(S)$ is a p -group by Lemma 2, $N_{k,\sigma}$ is a subgroup of $K(S)$. If $1 \leq k \leq r$, then $\{N_{k,\sigma}: 1 \leq k \leq r, \sigma \leq \nu\}$ is a smooth chain from $N_{r,0}$ to $N_{1,\nu} = K(S)$. The chain is ordered by the lexicographic order on $(-k, \sigma)$.

LEMMA 5. *Let $S \leq G$, S a finite p -group. If $N_{k,\sigma} \leq N \leq N_{k,\sigma+1}$, then every coset of N in $N_{k,\sigma+1}$ has an element proper in $V(G)$ with respect to N .*

PROOF. Fix $\alpha \in N_{k,\sigma+1}$. Let $\beta \in N$ and write $\alpha\beta = 1 + \sum_{i < \nu} g_i \alpha_i$ ($\alpha_i \in M^k$) in the unique representation above. Note for later use that $h(\alpha\beta) \leq h(g_\sigma \alpha_\sigma)$. Put $\beta' = \alpha^{-1}(1 + g_\sigma \alpha_\sigma)$. Then $\beta' \in N_{k,\sigma+1}$ since $1 + g_\sigma \alpha_\sigma \in N_{k,\sigma+1}$. Putting $\gamma = \sum_{i < \nu} g_i \alpha_i$, we have $\alpha\beta = 1 + \gamma$, where γ is nilpotent. Thus $(\alpha\beta)^{-1} = 1 - \gamma + \gamma^2 - \cdots = 1 - \gamma + \delta$, where $\delta \in \sum_{i < \nu} g_i M^{k+1}$. Consequently $\beta^{-1}\beta' = (1 - \gamma + \delta)(1 + g_\sigma \alpha_\sigma) = 1 - \gamma + g_\sigma \alpha_\sigma + \delta'$, where $\delta' \in \sum_{i < \nu} g_i M^{k+1}$. The g_σ term of $1 - \gamma$ is $-g_\sigma \alpha_\sigma$, hence $\beta^{-1}\beta' \in N_{k,\sigma} \subseteq N$. Therefore $\beta' \in N$. Moreover, $\alpha\beta' = 1 + g_\sigma \alpha_\sigma$, thus $h(\alpha\beta') = h(g_\sigma \alpha_\sigma) \geq h(\alpha\beta)$. Now we note that α_σ depends on β by writing it as $\alpha_\sigma(\beta)$. Thus we see that $\sup\{h(\alpha\beta): \beta \in N\} = \sup\{h(g_\sigma \alpha_\sigma(\beta)): \beta \in N\}$. But $g_\sigma \alpha_\sigma(\beta)$ has support in $g_\sigma S$, which is a finite set. Hence the supremum is achieved.

For S finite as above, FS is noetherian and hence M is nilpotent, say $M^{r+1} = 0$. Then $N_{r,0} = 1$ and $\{N_{k,\sigma}: 1 \leq k \leq r, \sigma \leq \nu\}$ is a smooth chain from 1 to $K(S)$. We may interpolate additional subgroups to obtain a composition series from 1 to $K(S)$. By the lemma, each subgroup is nice in the next, where heights are computed in $V(G)$. Under these circumstances, the relation "is nice in" is transitive, hence it follows that each subgroup is nice in $V(G)$ since $K(S)$ is nice by Lemma 4. Thus we have the

COROLLARY. *Let $S \leq G$, S a finite p -group. Then there is a nice composition series from 1 to $K(S)$ in $V(G)$.*

PROPOSITION 6. *If G is a simply presented p -group, then $V(G)$ is a simply presented p -group.*

PROOF. Let $\{N_\sigma: \sigma \leq \lambda\}$ be a nice composition series for G . By Lemma 4, $\{K(N_\sigma): \sigma \leq \lambda\}$ is a chain of nice subgroups of $V(G)$ from 1 to $V(G)$. Since an element of $V(G)$ has finite support, it is clear that it is a smooth chain. By Lemma 2, $V(G)/K(N_\sigma) = V(G/N_\sigma)$, and clearly $K(N_{\sigma+1})/K(N_\sigma) = K(N_{\sigma+1}/N_\sigma)$. By the corollary above, there is a nice composition series from 1 to $K(N_{\sigma+1}/N_\sigma)$, hence there is one from $K(N_\sigma)$ to $K(N_{\sigma+1})$. We thus obtain a nice composition series from 1 to $V(G)$, as desired.

4. Simply presented $V(G)$. Let $X \subseteq V(G)$ be a subset. We define the *support* $[X]$ of X to be the union of supports of the elements of X . We call X *saturated* if $[X] \subseteq X$. Thus X is saturated if and only if $X \cap G = [X]$. The usefulness of saturated subgroups is revealed by

LEMMA 7. *Let $N \leq V(G)$ be saturated. If N is nice in $V(G)$, then $N \cap G$ is nice in G .*

PROOF. Let $g \in G$. Then there exists $\alpha \in N$ such that $h(g\alpha)$ is maximum in gN . Choose g_1 in the support of α such that $h(g\alpha) = h(gg_1)$. Then $g_1 \in N \cap G$, and if $g_2 \in N \cap G$, we have $h(gg_2) \leq h(g\alpha) = h(gg_1)$. Therefore $N \cap G$ is nice in G .

Recall that a p -group is simply presented if and only if it has a nice system Γ . That is, Γ is a set of nice subgroups that contains the trivial subgroup, arbitrary products of members of Γ , and if $N \in \Gamma$ and X is a countable subset of the group, then there exists $N' \in \Gamma$ such that $N' \supseteq N \cup X$ and $(N' : N) \leq \aleph_0$.

LEMMA 8. *Let $V(G)$ have a nice system Γ . Suppose that $N \in \Gamma$ is saturated and that $X \subseteq V(G)$ is a countable subset. Then there exists $N' \in \Gamma$ such that N' is saturated, $N' \supseteq N \cup X$, and $(N' : N) \leq \aleph_0$.*

PROOF. Put $N_0 = N$. Choose $N_1 \in \Gamma$ such that $N_1 \supseteq N_0 \cup X$ and $(N_1 : N_0) \leq \aleph_0$. Note that $[N_0] \subseteq N_1$. Assume by induction that we have $N_0 \subseteq \cdots \subseteq N_k$ such that $N_i \in \Gamma$, $(N_i : N_0) \leq \aleph_0$ and $[N_{i-1}] \subseteq N_i$ ($1 \leq i \leq k$). Then $N_k = \langle N_{k-1}, Y \rangle$ for some countable set $Y \subseteq V(G)$. There exists $N_{k+1} \in \Gamma$ such that $N_{k+1} \supseteq N_k \cup [Y]$ and $(N_{k+1} : N_k) \leq \aleph_0$. But $[N_k] \subseteq [N_{k-1}] \cdot [Y] \subseteq N_{k+1}$, completing the induction. Put $N' = \bigcup_{i < \omega} N_i$. Then $N' \in \Gamma$, $N' \supseteq N \cup X$, $(N' : N) \leq \aleph_0$ and $[N'] = \bigcup_{i < \omega} [N_i] \subseteq N'$.

PROPOSITION 9. *If $V(G)$ is a simply presented p -group, then G is a simply presented p -group.*

PROOF. Choose a nice system Γ for $V(G)$ and put $\Gamma_0 = \{N \cap G : N \in \Gamma, N \text{ saturated}\}$. It suffices to show that Γ_0 is a nice system for G . In view of Lemmas 7 and 8, we only need show that Γ_0 is closed under arbitrary products. Therefore, suppose that $N_i \in \Gamma$, N_i saturated ($i \in I$), and consider $\prod_{i \in I} (N_i \cap G)$. Then $\prod (N_i \cap G) \subseteq (\prod N_i) \cap G \subseteq [\prod N_i] \subseteq \prod [N_i] = \prod (N_i \cap G)$. Thus, equality throughout implies that $\prod N_i \in \Gamma$ is saturated and that $\prod (N_i \cap G) = (\prod N_i) \cap G \in \Gamma_0$.

5. Proofs of the theorems. Theorem 1 follows immediately from Propositions 6 and 9. Thus, when G is simply presented one may describe the structure of $V(G)$ by calculating its Ulm invariants. This has been done for F a field by Mollov [12]. We now briefly describe these results when F is a perfect field of characteristic p . Let $f(\sigma)$ denote the σ th Ulm invariant of $V(G)$ and let λ denote the p -length of G .

- (1) If $G^\infty = 1$, then $f(\infty) = 0$; if $G^\infty \neq 1$, then $f(\infty) = |F| |G^\infty|$.
- (2) If $\lambda \leq \sigma < \infty$, then $f(\sigma) = 0$.
- (3) If $\sigma < \lambda$ and either $|F|$ or $|G^\sigma|$ is finite, then $f(\sigma) = |F| |G^\sigma|$.
- (4) If $\sigma < \lambda$ and both $|F|$ and $|G^\sigma|$ are finite, then

$$f(\sigma) = (|G^\sigma| - 2|G^{\sigma+1}| + |G^{\sigma+2}|) \log_p |F|.$$

Theorem 2 will follow from Theorem 1 by applying a simple lemma and trick used in [10]. The lemma [10, Lemma 2] asserts that G is a direct factor of $V(G)$ if G is

isomorphic to a direct factor of $V(G')$ for some group G' . Take $G' = \coprod_{i < \omega} G$. Then every nonzero Ulm invariant of G is multiplied by $\aleph_0$, and Lemma 3 implies that the Ulm invariants of G' do not exceed those of $V(G')$. Consequently, $G \times V(G')$ and $V(G')$ have the same Ulm invariants and are simply presented by Theorem 1. The lemma referred to now finishes the proof of Theorem 2.

The proof of Theorem 3 will require more work. We shall assume that G is a p -local Warfield group and that H is a group with $FG \cong FH$. It is well known that such an isomorphism may be assumed to preserve augmentation. We therefore write $FG = FH$ and think of H as a subgroup of $V = V(G) = V(H)$ that is a linear basis for FG . A number of conclusions about H now follow from Lemmas 1 and 2. Since tG is a p -group, we see that tH is a p -group. Now let $q \neq p$ be a prime. Since G is q -divisible, the same follows for V . But $H/tH \cong V/K(tH)$, thus H/tH is q -divisible, and hence so is H . Therefore H is a p -local group which we proceed to show is a Warfield group.

Let X' be a nice decomposition basis of G (see [7]). Since V/H is torsion by Lemma 2, there exists a subordinate basis X contained in H . But subordinates of nice decomposition bases remain nice decomposition bases. By Lemma 4, since G and H are isotype in V with torsion quotients, we conclude that X is a nice decomposition basis of H as well as G . Therefore G and H have the same Warfield invariants, and, as remarked in the introduction, they are known to have the same Ulm invariants. Thus we will be done if we finish showing that H is a Warfield group. Let $B = \langle X \rangle$. Then $F(G/B) = FG/FG \cdot I(B) = FH/FH \cdot I(B) = F(H/B)$. But G a Warfield group implies that G/B is a simply presented p -group. Theorem 1 implies H/B is simply presented, therefore H is a Warfield group.

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