FACTORIZATIONS OF LEBESGUE MEASURE VIA CONVOLUTIONS

NORMAN LEVENBERG, JR., GAVEN J. MARTIN, ALLEN L. SHIELDS AND
 SMILKA ZDRAVKOVSKA

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ABSTRACT. Given a continuous, increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$, we define the Hausdorff ϕ -measure of a bounded set E in the unit interval $I = [0, 1]$ as $H_\phi(E) = \lim_{\delta \rightarrow 0} H_{\phi, \delta}(E)$ where $H_{\phi, \delta}(E) = \inf \sum_{i=1}^{\infty} \phi(t_i)$ and the infimum is taken over all countable covers of E by intervals U_i with $t_i = |U_i| = \text{length of } U_i < \delta$. We show that given any such ϕ , there exist closed, nowhere dense sets $E_1, E_2 \subset I$ with $H_\phi(E_1) = H_\phi(E_2) = 0$ and $E_1 + E_2 \equiv \{a + b: a \in E_1, b \in E_2\} = I$. The sets E_i ($i = 1, 2$) are constructed as Cantor-type sets $E_i = \bigcap_{n=1}^{\infty} E_{i,n}$ where $E_{i,n}$ is a finite union of disjoint closed intervals. In addition, we give a simple geometric proof that the natural probability measures μ_i supported on E_i which arise as weak limits of normalized Lebesgue measure on $E_{i,n}$ have the property that the convolution $\mu_1 * \mu_2$ is Lebesgue measure on I .

Introduction. In 1949, H. G. Eggleston [E] showed that there exist compact subsets E_1 and E_2 of the unit interval $I = [0, 1]$ which have Hausdorff dimension zero, but nonetheless have the property that the algebraic sum $E_1 + E_2 = \{x + y: x \in E_1 \text{ and } y \in E_2\}$ is the entire unit interval I .

Following the ideas of Eggleston, we generalize his result by showing that for any given continuous increasing function Φ with $\Phi(0) = 0$, there are compact subsets E_1 and E_2 of the unit interval I such that $E_1 + E_2 = I$ and the Hausdorff Φ -measure of both E_1 and E_2 is zero.

In addition we show that the natural probability measures μ_1 and μ_2 , supported on E_1 and E_2 , are measures whose convolution $\mu_1 * \mu_2$ is Lebesgue measure λ on I . Thus $\mu_1 * \mu_2 = \lambda$ can be thought of as a nontrivial factorization of Lebesgue measure. In the last section we will consider such factorizations of other measures absolutely continuous with respect to Lebesgue measure.

We wish to thank R. Lyons for pointing out a probabilistic approach to the proof of Theorem 3.1, which we outline. We would also like to acknowledge that W. Rudin communicated to the first named author an independent proof of Theorem 3.1 and some of its consequences. It was also pointed out to us that a similar question was studied by Tortrat who obtained a version of Theorem 3.1; see [T].

Finally, we wish to thank the referee who pointed out an error in an earlier draft and provided us with some additional references.

1. We begin with the general construction of the sets E_1 and E_2 . Given a sequence of positive integers greater than one $\{q_i\}_{i=1,2,\dots}$ (by convention we set

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$q_0 = 1$) we can express any number t in the unit interval $I = [0, 1]$ in the form

$$(1.1) \quad t = \sum_{i=1}^{\infty} p_i / (q_1 \cdot q_2 \cdots q_i),$$

where $\{p_i\}_{i=1,2,\dots}$ is a sequence of integers satisfying $0 \leq p_i < q_i$.

One can see this fact geometrically by subdividing the interval into q_1 equal subintervals and each of those subintervals into q_2 further subintervals and so on. The number p_1 tells which of the first q_1 intervals the point lies, p_2 tells which further subinterval the point lies and so on (there is only an ambiguity at the endpoints of each subinterval and this is easily dealt with).

Let

$$E_1 = \left\{ x \in I : x = \sum_{i=1}^{\infty} p_{2i-1} / (q_1 \cdot q_2 \cdots q_{2i-1}) \right\}$$

and

$$E_2 = \left\{ x \in I : x = \sum_{i=1}^{\infty} p_{2i} / (q_1 \cdot q_2 \cdots q_{2i}) \right\}.$$

We make the following observations about the sets E_1 and E_2 .

(A) $E_1 + E_2 = I$.

(B) The sets E_1 and E_2 are perfect, nowhere dense sets that may be written as

$$E_1 = \bigcap_{n=1}^{\infty} E_{1,n}, \quad E_2 = \bigcap_{n=1}^{\infty} E_{2,n},$$

where $E_{1,1} \supseteq E_{1,2} \supseteq E_{1,3} \supseteq \cdots$ and $E_{2,1} \supseteq E_{2,2} \supseteq E_{2,3} \supseteq \cdots$ are a nested sequence of closed sets and $E_{1,n}$ consists of $c_n = q_1 \cdot q_3 \cdots q_{2n-1}$ disjoint intervals of length $1/(q_1 \cdot q_2 \cdots q_{2n})$, and $E_{2,n}$ consists of $d_n = q_2 \cdot q_4 \cdots q_{2n}$ disjoint intervals of length $1/(q_1 \cdot q_2 \cdots q_{2n+1})$.

It is not difficult to see that E_i , $i = 1, 2$, is perfect (we may perturb the expansion to find a sequence converging to any given point). To simplify notation we set $r_n = q_1 \cdot q_2 \cdots q_n$. To see (B), we inductively construct the sets $E_{i,n}$. We construct $E_{1,1}$ by dividing I into q_1 equal intervals of length $1/q_1$; $E_{1,1}$ is then

$$E_{1,1} = \bigcup \{ [j/q_1, j/q_1 + 1/q_1 \cdot q_2] : j = 0, 1, 2, \dots, q_1 - 1 \}.$$

Given $E_{1,n}$, we construct $E_{1,n+1}$ by subdividing each subinterval I_n of $E_{1,n}$ (there are c_n such intervals of length $1/r_{2n}$) into q_{2n+1} intervals of equal length and from each such interval we choose the leftmost subinterval I_{n+1} of length $1/r_{2n+2}$. The union of all the intervals so chosen is then $E_{1,n+1}$ (see Figure 1).

We construct $E_{2,n}$ in a similar fashion. We first subdivide the interval $[0, 1/q_1]$ into q_2 subintervals of equal length $1/(q_1 \cdot q_2)$; $E_{2,1}$ is then the union of q_2 closed subintervals

$$E_{2,1} = \bigcup \{ [j/r_2, j/r_2 + 1/r_3] : j = 0, 1, 2, \dots, q_2 - 1 \}.$$

Then given $E_{2,n}$ we construct $E_{2,n+1}$ by subdividing each subinterval of $E_{2,n}$ (there are d_n such subintervals each of length $1/r_{2n+1}$) into q_{2n+1} intervals of

FIGURE 1

equal length and then choosing the leftmost subinterval of each such interval with length $1/r_{2n+3}$. The union of all the intervals so chosen is $E_{2,n+1}$.

It is now clear that $\{E_{i,n}\}_{n=1,2,\dots}$ is a closed and nested sequence of sets. We set $F_i = \bigcap E_{i,n}$. We wish to show that F_i is a closed, perfect, nowhere dense set and that $F_i = E_i$.

It is clear that F_i is closed. Since the diameter of each component of $E_{i,n}$ has length no more than $1/r_{2n}$, we see that the diameter of each component of F_i is zero. That is, F_i is totally disconnected; and since it is closed, it is nowhere dense. Thus if we show $E_i = F_i$ we will have proved (B). (A) follows from the choice of E_1 and E_2 .

We show that $E_1 = F_1$; $E_2 = F_2$ will follow in a similar fashion. The set of left endpoints L_n of the intervals of $E_{1,n}$ is

$$L_n = \left\{ x = \sum_{i=1}^n p_{2i-1}/r_{2i-1} : 0 \leq p_{2i-1} < q_{2i-1} \right\}.$$

Thus L_n lies in E_1 for all n . Hence $\bigcup L_n$ is contained in E_1 . Since $\bigcup L_n$ is dense in F_1 and since E_1 is closed, we find F_1 is contained in E_1 . Similarly, since the set of left endpoints is retained at each stage, $F_1 \supseteq \dots \supseteq L_{n+1} \supseteq L_n$, and since each point of E_1 is a limit of points of $\bigcup L_n$, we find E_1 is contained in F_1 , since F_1 is closed. Hence $E_1 = F_1$.

We now define probability measures $\mu_{i,n}$ supported on $E_{i,n}$, $i = 1, 2$, by using a constant multiple of Lebesgue measure on each subinterval $I_{i,n}$ of $E_{i,n}$. More precisely, we let $\chi_{i,n}$ be the characteristic function of $E_{i,n}$ and set

$$(1.2) \quad d\mu_{1,n} = q_2 \cdot q_4 \cdots q_{2n} \cdot \chi_{1,n}(x) d\lambda(x),$$

$$(1.3) \quad d\mu_{2,n} = q_1 \cdot q_3 \cdots q_{2n+1} \cdot \chi_{2,n}(x) d\lambda(x).$$

From our construction of the sets $E_{i,n}$ we find that the measures $\mu_{i,n}$ have the following properties:

$$(C) \quad \mu_{i,n}([0, 1]) = 1, \quad n = 1, 2, \dots;$$

$$(D) \quad \mu_{i,n}(I_{i,n}) = \mu_{i,m}(I_{i,n}) \quad \text{for } m \geq n;$$

(E) the measures $\{\mu_{i,n}\}_{n=1,2,\dots}$ converge weak* to a probability measure μ_i supported on E_i .

(D) can be rephrased by saying that the intervals $I_{i,n}$ carry the same mass from some finite stage onwards. This fact will ensure the weak* convergence of the measures indicated in (E).

Henceforth, E_i and μ_i will refer to the sets and measures obtained above from a given sequence $\{q_n\}$. We will show that an appropriate choice of the sequence $\{q_n\}$ we lead to some interesting consequences. See also [KS] for other related Cantor set constructions.

2. Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be any continuous increasing function with $\Phi(0) = 0$. Given a bounded subset E of the interval I , we define

$$H_{\Phi,\delta}(E) = \inf \sum_{i=1}^{\infty} \Phi(t_i),$$

where the infimum is taken over all countable covers of E by intervals U_i each of length $t_i = \lambda(U_i) < \delta$. Clearly $H_{\Phi,\delta}(E) \geq H_{\Phi,\delta'}(E)$ if $\delta \leq \delta'$, and hence we may define

$$H_{\Phi}(E) = \lim_{\delta \rightarrow 0} H_{\Phi,\delta}(E)$$

and call $H_{\Phi}(E)$ the Φ -measure of E . For $\Phi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, we obtain the classical α -Hausdorff measure, $H_{\alpha}(E)$. If $H_{\alpha}(E) = 0$ for $\alpha > \beta$ and $H_{\alpha}(E) = +\infty$ for $\alpha < \beta$, we say E has Hausdorff dimension β . We now strengthen Eggleston's result with the following theorem.

2.1 THEOREM. *Given any continuous increasing function Φ with $\Phi(0) = 0$, we can construct subsets E_1 and E_2 of the unit interval such that*

$$(2.2) \quad H_{\Phi}(E_1) = H_{\Phi}(E_2) = 0$$

and

$$(2.3) \quad E_1 + E_2 = I.$$

PROOF. We construct a sequence $\{q_k\}$ inductively as follows. Set $q_0 = 1$. Given $q_0, q_1, \dots, q_n$ we choose q_{n+1} so large that

$$(2.4) \quad \Phi(1/q_{n+1}) < 1/(n \cdot r_n).$$

To show that $H_{\Phi}(E_1) = 0$, it suffices to show that there is a sequence of positive numbers $\{\delta_n\}$, such that $\delta_n \rightarrow 0$ and that

$$(2.5) \quad H_{\Phi,\delta_n}(E_1) = O(1/n).$$

Since $E_{1,n}$ consists of $q_1 \cdot q_3 \cdots q_{2n-1}$ intervals $I_{1,n}$ of length $1/(q_1 \cdot q_2 \cdots q_{2n}) = \delta_n$, and since E_1 is contained in $E_{1,n}$, we see that

$$\begin{aligned} H_{\Phi,\delta_n}(E_1) &\leq H_{\Phi,\delta_n}(E_{1,n}) \leq q_1 \cdot q_3 \cdots q_{2n-1} \Phi(1/(q_1 \cdot q_2 \cdots q_{2n})) \\ &\leq q_1 \cdot q_3 \cdots q_{2n-1} \Phi(1/q_{2n}) \quad (\text{as } \Phi \text{ is increasing}) \\ &\leq q_1 \cdot q_3 \cdots q_{2n-1} 1/((2n-1)q_1 \cdot q_2 \cdots q_{2n-1}) \\ &\leq 1/(2n-1). \end{aligned}$$

A similar calculation shows that $H_{\Phi}(E_2) = 0$. The fact that $E_1 + E_2 = I$ has been noted above. The proof is complete.

For a subset E of the unit interval we define the logarithmic measure of E as $L(E) = H_{\Phi}(E)$, where $\Phi(t) = 1/\log(1/t)$.

2.6 COROLLARY. *There exist compact subsets E_1 and E_2 of the unit interval satisfying*

$$(2.7) \quad L(E_1) = L(E_2) = 0$$

and

$$(2.8) \quad E_1 + E_2 = I.$$

In particular, $L(E_1 + E_2) = \infty$.

It is known (see [N, Chapter 5, §6, p. 147]) that a set of logarithmic measure zero has logarithmic capacity zero. Thus Corollary 2.6 implies the existence of two compact sets E_1 and E_2 each with capacity zero and yet their algebraic sum, which is the whole interval, has positive capacity.

2.7 PROPOSITION. *The measure μ_1 associated with the set E_1 in Corollary 2.6 is an Evans measure for the set E_1 . That is, the potential function*

$$P\mu_1(z) = \int_{E_1} \log(1/|z - y|) d\mu_1(y)$$

satisfies

$$(2.8) \quad P\mu_1(z) = +\infty \quad \text{for } z \in E_1,$$

$$(2.9) \quad P\mu_1(z) < +\infty \quad \text{for } z \in \mathbf{R}^2 - E_1.$$

Similarly, the measure μ_2 is an Evans measure for the set E_2 .

PROOF. Since μ_1 is a probability measure supported on the compact set E_1 , (2.9) is immediate. To prove (2.8) we fix an $x \in E_1$. In our construction of the sets E_1 and E_2 in Corollary 2.6 we chose integers $q_1, q_2, \dots$ satisfying (2.4) for $\Phi(t) = 1/\log(1/t)$. Thus

$$(2.10) \quad \log(q_{n+1}) > n \cdot q_1 \cdot q_2 \cdots q_n.$$

Then for each n , there is a subinterval $I_{1,n}$ of $E_{1,n}$ which contains x . Now $0 \leq x, y \leq 1$ implies that $\log(1/|x - y|) \geq 0$ and so

$$\begin{aligned} P\mu_1(x) &\geq \int_{I_{1,n}} \log(1/|x - y|) d\mu_1(y) \\ &\geq \log(1/\lambda(I_{1,n}))\mu_1(I_{1,n}) \\ &= \log(1/\lambda(I_{1,n}))\mu_{1,n}(I_{1,n}) \quad \text{by (C) and (D)} \\ &= (\log(q_1 \cdot q_2 \cdots q_{2n})) / (q_1 \cdot q_3 \cdots q_{2n-1}) \\ &\geq \log(q_{2n}) / (q_1 \cdot q_3 \cdots q_{2n-1}) \\ &\geq 2n - 1 \quad \text{by (2.10).} \end{aligned}$$

The result now follows.

3. We have seen that by choosing our sequence $\{q_n\}$ so that the q_n 's increase sufficiently rapidly, we can construct very small sets E_1 and E_2 whose algebraic sum is the whole interval I . The convolution $\mu_1 * \mu_2$ of the measures μ_1 and μ_2 is supported in the algebraic sum of the supports of μ_1 and μ_2 , and thus in the whole interval. We now show that no matter how small the sets E_i are, the measure $\mu_1 * \mu_2$ is large.

3.1 THEOREM. For any sequence $\{q_n\}_{n=1,2,\dots}$ with $1 < q_1 \leq q_2 \leq \dots$ we have

$$(3.2) \quad \mu_1 * \mu_2 = \lambda \quad (= \text{Lebesgue measure on } I),$$

$$(3.3) \quad \mu_{1,n} * \mu_{2,m} = \lambda \pmod{1} \quad \text{for any pair of positive integers } m \text{ and } n.$$

PROOF. Here $\text{mod } 1$ refers to the fact that the support of $\mu_{1,n} * \mu_{2,m}$ will extend slightly beyond the interval I ; if we add this extra mass starting at the origin, we do indeed obtain Lebesgue measure. This problem could be circumvented by working on the unit circle, but this would then complicate the geometrically simple proof.

That (3.2) follows from (3.3) can be seen from the fact that $\mu_{i,n} \rightarrow \mu_i$ weak* and thus $(\mu_{1,n} \times \mu_{2,n}) \rightarrow (\mu_1 \times \mu_2)$ weak* on the product space $E_1 \times E_2$, together with the fact that the convolution is obtained by integrating on the product space. That is,

$$\int f d(\mu_1 * \mu_2) = \iint f(x+y) d\mu_1(x) d\mu_2(y).$$

We will prove that $\mu_{1,n} * \mu_{2,n} = \lambda$. The general case $\mu_{1,n} * \mu_{2,m} = \lambda$ will then follow from a simple geometric observation. We will outline the proof in the case that $n = m = 1$. This case is typical of the general case and its outline will simplify and motivate the general discussion following.

The measure $\mu_{1,1}$ is supported on the q_1 intervals $I_{1,1}$ comprising $E_{1,1}$ and the measure $\mu_{2,1}$ is supported on the q_2 intervals $I_{2,1}$ comprising $E_{2,1}$. Hence the product measure $\mu_{1,1} \times \mu_{2,1}$ is supported on $q_1 \cdot q_2$ rectangles which we denote by $R_{i,j}$. Also

$$(\mu_{1,1} \times \mu_{2,1})(R_{i,j}) = \mu_{1,1}(I_{1,1}) \cdot \mu_{2,1}(I_{2,1}) = 1/(q_1 \cdot q_2).$$

We illustrate this in Figure 2 by putting $E_{1,1}$ on the vertical axis and $E_{2,1}$ on the horizontal axis.

We define $p(x, y) = x + y$. The diagonal lines in Figure 2 are to indicate the effect of projection via p . We compute the $\mu_{1,1} * \mu_{2,1}$ measure of a Borel subset E of I as

$$(\mu_{1,1} * \mu_{2,1})(E) = (\mu_{1,1} \times \mu_{2,1})(p^{-1}(E)).$$

The idea of the proof that $\mu_{1,1} * \mu_{2,1} = \lambda$ is that the rectangles project down to exactly fill out the interval I . We compute the $\mu_{1,1} * \mu_{2,1}$ measure of E as the sum

$$\mu_{1,1} * \mu_{2,1}(E) = \sum_{i,j} q_1 \cdot q_2 \cdot q_3 \cdot (\lambda \times \lambda)(p^{-1}(E) \cap R_{i,j}).$$

Now projecting the rectangles down the diagonal so that their bases lie on the horizontal axis will not change the two-dimensional Lebesgue measure of $p^{-1}(E) \cap R_{i,j}$. After such projection, the rectangles will fill out the strip $Q_1 = [0, 1] \times [0, 1/(q_1 \cdot q_2 \cdot q_3)]$. Then we find that the measure

$$(3.4) \quad \begin{aligned} \mu_{1,1} * \mu_{2,1}(E) &= q_1 \cdot q_2 \cdot q_3 \cdot (\lambda \times \lambda)(p^{-1}(E) \cap Q_1) \\ &= r_3 \cdot \lambda(E) \cdot (1/r_3) \pmod{1} \\ &= \lambda(E) \pmod{1}. \end{aligned}$$

We need some notation and definitions to formalize the above discussion. For a rectangle $R = [a, b] \times [c, d]$ in $[0, 1] \times [0, 1]$ we define the projection along the diagonal $P(R)$ to be the rectangle

$$P(R) = [a + c, b + c] \times [0, d - c] \pmod{1},$$

FIGURE 2

where again the mod 1 refers to the addition of the overlap to the origin.

We will consider the case $\mu_{1,n} * \mu_{2,n}$. Let R be a rectangle in the support of $\mu_{1,n} \times \mu_{2,n}$. Then $R = I_{1,n} \times I_{2,n}$ for appropriate subintervals $I_{1,n}$ of $E_{1,n}$ and $I_{2,n}$ of $E_{2,n}$. R is isometric to the rectangle

$$[0, 1/(q_1 \cdot q_2 \cdots q_{2n})] \times [0, 1/(q_1 \cdot q_2 \cdots q_{2n} \cdot q_{2n+1})] = [0, 1/r_{2n}] \times [0, 1/r_{2n+1}]$$

and has total $(\mu_{1,n} \times \mu_{2,n})$ -mass $1/r_{2n}$. We set $Q_n = [0, 1] \times [0, 1/r_{2n+1}]$. We wish to show that the projection of all the rectangles precisely fills out Q_n with no overlap of interior points. This will then prove (3.3) for the case $m = n$. We index all the rectangles in the support of $\mu_{1,n} \times \mu_{2,n}$ as $\{R^{n_k}\}_{k=1,2,\dots}$.

3.5 PROPOSITION. For each $n = 0, 1, 2, \dots$,

$$(3.6) \quad \bigcup \{P(R^{n_k}) : k = 1, 2, \dots, r_{2n}\} = Q_n.$$

Furthermore,

$$(3.7) \quad \text{int}(P(R^{n_k})) \cap \text{int}(P(R^{n_j})) = \emptyset, \quad k \neq j.$$

PROOF. We use induction. Since $q_0 = 1$ and $R^{0_0} = I \times [0, q_1] = P(R^{0_0}) = Q_0$, the case $n = 0$ is immediate. We assume that (3.6) and (3.7) are both true for the index n . To prove (3.6) and (3.7) for the index $n + 1$, it suffices to show that for each rectangle R^{n_k} , the projection of the subrectangles obtained from the $(n + 1)$ st subdivision which lie in the rectangle R^{n_k} cover the base of the projection of R^{n_k} without overlap. More precisely, we wish to show that

$$(3.8) \quad \text{if } P(R^{n_k}) = [a, a + 1/r_{2n}] \times [0, 1/r_{2n+1}], \\ \text{then } \bigcup \{P(R^{n_{k+1_j}}) : R^{n_k} \supseteq R^{n_{k+1_j}}\} = [a, a + 1/r_{2n}] \times [0, 1/r_{2n+1}].$$

One can see that this really is identical to the case occurring in the first subdivision, except that it is on a smaller rectangle. It is clear that we may assume for simplicity that $a = 0$. Thus we set W as the rectangle $W = [0, 1/r_{2n}] \times [0, 1/r_{2n+1}]$.

One can think of W as the large rectangle of Figure 2. In the construction of the $(n+1)$ th stage, we subdivide the interval $[0, 1/r_{2n}]$ into q_{2n+1} intervals and take the leftmost subinterval of length $1/r_{2n+2}$ of each such interval. Similarly we subdivide $[0, 1/r_{2n+1}]$. Thus we obtain q_{2n+1} stacks of q_{2n+2} rectangles. A typical rectangle in the j th stack ($j = 0, 1, 2, \dots, q_{2n+1} - 1$) is of the form

$$B_{j,k} = [j/r_{2n+1}, j/r_{2n+1} + 1/r_{2n+2}] \times [k/r_{2n+2}, k/r_{2n+2} + 1/r_{2n+3}], \\ k = 0, 1, 2, \dots, q_{2n+2} - 1.$$

Then

$$P(B_{j,k}) = [j/r_{2n+1} + k/r_{2n+2}, j/r_{2n+1} + (k+1)/r_{2n+2}] \times [0, 1/r_{2n+3}],$$

and we see that the $P(B_{j,k})$ are nonoverlapping. Furthermore, we find

$$\bigcup \{P(B_{j,k}) : k = 0, 1, \dots, q_{2n+2} - 1\} = [j/r_{2n+1}, (j+1)/r_{2n+1}] \times [0, 1/r_{2n+3}],$$

and it then follows that

$$\bigcup \{P(B_{j,k}) : k = 0, 1, \dots, q_{2n+2} - 1, j = 0, 1, \dots, q_{2n+1} - 1\} = W.$$

This then establishes the proposition.

From the above, as in (3.4), we can now show that $\mu_{1,n} * \mu_{2,n} = \lambda$ for

$$(3.9) \quad (\mu_{1,n} * \mu_{2,n})(E) = (\mu_{1,n} \times \mu_{2,n})(p^{-1}(E)) = r_{2n+1}(\lambda \times \lambda)(p^{-1}(E) \cap Q_n) \\ = r_{2n+1} \cdot \lambda(E) \cdot (1/r_{2n+1}) = \lambda(E) \pmod{1}. \quad \square$$

Suppose now that $m \geq n$. Then the support of $\mu_{1,n} \times \mu_{2,m}$ lies in the support of $\mu_{1,n} \times \mu_{2,n}$ and is obtained by successively subdividing the intervals $I_{1,n}$ on the vertical axis. Thus we subdivide each rectangle into horizontal strips. The projection of these strips will be several horizontal strips $S_{n,m}$ in Q_n (since each rectangle projects into Q_n). The mass of each strip is the same and their total mass is one (the $\mu_{i,n}$ are probability measures). One then easily sees that

$$(3.10) \quad \mu_{1,n} * \mu_{2,m}(E) = (\mu_{1,n} \times \mu_{2,m})\left(p^{-1}(E) \cap \left(\bigcup S_{n,m}\right)\right) \\ = (\mu_{1,n} \times \mu_{2,n})(p^{-1}(E) \cap Q_n) = \lambda(E) \quad \text{by (3.9).}$$

Finally, for the case $m \leq n$, we can project on to the vertical axis and use a similar argument. This completes the proof of Theorem 3.1.

Combining Corollary 2.6, Proposition 2.7 and Theorem 3.1, we obtain the following

3.11 THEOREM. *There are compact subsets E_1 and E_2 of the unit interval, each of capacity zero, such that the convolution of the natural Evans measures μ_1 and μ_2 supported on the sets E_1 and E_2 is Lebesgue measure.*

It is worth remarking that the measures μ_1 and μ_2 both have infinite energy, while their convolution is Lebesgue measure on I and so has finite energy. Also it is not difficult to see that this result is true in all dimensions. For instance,

$$(E_1 \times E_1 \times \cdots \times E_1) + (E_2 \times E_2 \times \cdots \times E_2) = I^n,$$

and for a given Φ the Hausdorff Φ -measure of $E_i \times E_i \times \cdots \times E_i$ can be made arbitrarily small by assuring that E_i is sufficiently small, $i = 1, 2$. That the natural

probability measures supported on $E_i \times E_i \times \cdots \times E_i$, $i = 1, 2$, convolve to give the usual Lebesgue measure of the cube I^n is not difficult and we leave the details to the reader.

An alternative proof of Theorem 3.1, using probability theory, has been pointed out to us by R. Lyons. We omit the technical details and briefly outline the method here. One can show that

$$(3.12) \quad \lambda = \bigstar_{i=1}^{\infty} S_i, \quad \text{where } S_i = \sum_{p=0}^{q_i-1} \left(\frac{1}{q_i} \right) \delta \left(\frac{p}{r_i} \right).$$

To see this, we consider the successive p_i in the expansion of a point x (see (1.1)) to be a type of decimal expansion. We then attach equal mass at each possible point of such an expansion. A probabilistic argument then shows that this is indeed Lebesgue measure. Similarly, we find

$$(3.13) \quad \mu_{1,n} = \left(\bigstar_{1 \leq k \leq 2n+1}^{k \text{ odd}} S_k \right) * \left(\bigstar_{j > 2n+1}^{\infty} S_j \right),$$

$$(3.14) \quad \mu_{2,n} = \left(\bigstar_{2 \leq k \leq 2n+2}^{k \text{ even}} S_k \right) * \left(\bigstar_{j > 2n+2}^{\infty} S_j \right).$$

This is seen in much the same way as (3.12). Now it is true that all the infinite convolutions here converge unconditionally. Hence in the convolution $\mu_{1,n} * \mu_{2,m}$ we may rearrange terms to obtain Lebesgue measure (as in (3.12)) convolved with an error term from the extra factors, see, for instance, Chapter 6 of [GM]. However, convolving any measure with Lebesgue measure (on the circle) yields Lebesgue measure and so we find that $\mu_{1,n} * \mu_{2,m} = \lambda \pmod{1}$ as above.

4. In the last section we found a nontrivial factorization of Lebesgue measure. This is, neither of the measures μ_1 nor μ_2 are point masses. We now consider what other measures could possibly be factorized in such a fashion and how Lebesgue measure can be determined from the measures μ_1 and μ_2 . Throughout this section we fix two sets E_1 and E_2 as constructed in §1.

Given a nonnegative Lebesgue integrable function f defined on the unit interval with $\int f d\lambda = 1$, we obtain a probability measure $f(x) d\lambda(x)$. Suppose we could factor $f d\lambda$ as

$$(4.1) \quad f(x) d\lambda(x) = \eta_1(x) * \eta_2(x),$$

where η_1 and η_2 are probability measures supported on the sets E_1 and E_2 , respectively.

4.2 PROPOSITION. *With the above assumptions, there is a μ_1 -measurable function g such that*

$$d\eta_1(x) = g(x) d\mu_1(x)$$

and a μ_2 -measurable function h such that

$$d\eta_2(x) = h(x) d\mu_2(x).$$

PROOF. Equation (4.1) implies that for any Lebesgue measurable set F contained in the interval I ,

$$(4.3) \quad \lambda_f(F) = \int_F f(t) d\lambda(t) = \iint_{E_1 \times E_2} \chi_F(x+y) d\eta_1(x) d\eta_2(y).$$

We recall that $E_1 + E_2 = I$. Given two Borel subsets F_1 and F_2 of E_1 and E_2 we set $F = F_1 + F_2$. It easily follows that

$$(4.4) \quad \chi_F(x+y) = \chi_{F_1}(x) \cdot \chi_{F_2}(y) = \chi_{(F_1 \times F_2)}(x, y) \quad \text{for } x \in E_1 \text{ and } y \in E_2.$$

The above, together with (4.3) and Fubini's Theorem, yield

$$(4.5) \quad \lambda_f(F) = \eta_1(F_1) \cdot \eta_2(F_2),$$

and consequently

$$(4.6) \quad \eta_1(F_1) = \lambda_f(F_1 + E_2) \quad \text{and} \quad \eta_2(F_2) = \lambda_f(F_2 + E_1).$$

We now define

$$g(x) = \int_{E_2} f(x+y) d\mu_2(y) \quad \text{for } x \in E_1$$

and

$$h(y) = \int_{E_1} f(x+y) d\mu_1(x) \quad \text{for } y \in E_2.$$

By Fubini's Theorem, g is μ_1 -measurable and h is μ_2 -measurable. Furthermore, we find from the above (recalling $\mu_1 * \mu_2 = \lambda$)

$$(4.7) \quad \begin{aligned} \eta_1(F_1) &= \lambda_f(F_1 + E_2) = \iint_{E_1 \times E_2} \chi_{F_1}(x) f(x+y) d\mu_1(x) d\mu_2(y) \\ &= \int_{F_1} g(x) d\mu_1(x). \end{aligned}$$

Similarly we find

$$\eta_2(F_2) = \int_{F_2} h(y) d\mu_2(y)$$

This completes the proof.

Next Proposition 4.2 and (4.3) yield

$$(4.8) \quad \begin{aligned} &\iint_{E_1 \times E_2} \chi_F(x+y) f(x+y) d\mu_1(x) d\mu_2(y) \\ &= \iint_{E_1 \times E_2} \chi_F(x+y) g(x) h(y) d\mu_1(x) d\mu_2(y) \end{aligned}$$

for all Borel subsets F of I .

It is not difficult to see that (4.4) implies that continuous functions of the form $f(x+y)$ are dense in the space of continuous functions on $E_1 \times E_2$. Consequently from (4.8) we find

$$(4.9) \quad f(x+y) = g(x)h(y), \quad (x, y) \in E_1 \times E_2 \text{ a.e. w.r.t. } \mu_1 \times \mu_2.$$

At a point (a, b) where (4.9) is satisfied we find

$$(4.10) \quad f(y+a) = g(a)h(y) \text{ a.e. } \mu_2 \quad \text{and} \quad f(x+b) = h(b)g(x) \text{ a.e. } \mu_1.$$

Thus g and h are determined by the values of f on the sets E_1 and E_2 . Let us assume for the sake of simplicity that $(a, b) = (0, 0)$ and that $f(0) = g(0) = h(0) = 1$. Then (4.9) becomes

$$(4.11) \quad f(x + y) = f(x)f(y) \quad \text{a.e. } \mu_1 \times \mu_2.$$

If f is a differentiable function of the interval, then (4.11) easily yields $f(t) = ce^{dt}$. Equation (4.11) is indeed very restrictive, in general, and leads one to suspect that there are very few measures of the form $f d\lambda$ which can be factored via the convolution of two such measures.

One can actually find solutions to (4.11) which are continuous except at a countable number of points as follows:

Let g and h be continuous functions on E_1 and E_2 , respectively (where the topology of E_1 and E_2 is the topology they inherit as subspaces of $[0, 1]$). We would like to define a continuous function f on the interval I , by the rule

$$(4.12) \quad f(t) = g(x) \cdot h(y),$$

where $t = x + y$, $x \in E_1$ and $y \in E_2$.

Thinking of the expansion of a point $t \in I$ (see (1.1)) as a "decimal" expansion $t = (0.p_1p_2p_3\dots)$, we see by definition the sets E_1 and E_2 are the points with expansion

$$x \in E_1 \Rightarrow x = (0.p_10p_30p_50\dots), \quad \text{"zeros in the even slots"},$$

and

$$y \in E_2 \Rightarrow y = (0.0p_20p_40\dots), \quad \text{"zeros in the odd slots"}.$$

This gives us two functions $t \rightarrow x(t)$ and $t \rightarrow y(t)$ defined by the above splitting of the expansion. However, the expansion of a point $t \in I$ need not be unique (as in the case of repeating 9's in a decimal expansion) so that a priori the functions $x(t)$ and $y(t)$ are not well defined. We can alleviate this problem by identifying the points

$$(0.p_1p_2\dots p_n(q_{n+1} - 1)(q_{n+2} - 1)\dots) \quad \text{and} \quad (0.p_1p_2\dots (p_n + 1)000\dots),$$

where p_n is the last integer in the expansion with $p_i < q_i - 1$, and always agree to choose the latter expansion for such a point t . This then makes the expansion unique and the functions $x(t)$ and $y(t)$ well defined on I . Using the rule suggested in (4.12) we can define a function on the interval as

$$f(t) = g(x(t)) \cdot h(y(t)).$$

The function f will be continuous except at a countable number of points provided we show that the functions $x(t)$ and $y(t)$ are. We will show that $x(t)$ is continuous except at the points with expansion terminating in zeros. That $y(t)$ is continuous except at such points will follow in a similar fashion. Thus let t be a point of I whose expansion does not terminate in zeros and let $\delta > 0$. Let N be so large that $1/r_N < \delta$. Next, by our choice of the expansion of t , there is an $i > N$ such that $0 < p_i < q_i - 1$. We accordingly set $\tau = 1/r_i$. We now see that if $|t - t'| < \tau$, then the expansions of t and t' agree to the first $i - 1$ places and hence so do the expansions of $x(t)$ and $x(t')$. Thus

$$|x(t) - x(t')| < 1/r_{i-1} \leq 1/r_N < \delta.$$

It is not difficult to see that the functions $x(t)$ and $y(t)$ are not continuous at points whose expansion terminates in zeros since a small perturbation will carry far up the expansion, and so change $x(t)$ quite a bit.

The following result follows easily from the above.

4.13 THEOREM. *If f is a nonnegative continuous function on I , then*

$$f d\lambda = \eta_1 * \eta_2,$$

where η_1 and η_2 are measures supported on E_1 and E_2 , respectively, only if there are continuous functions g and h defined on E_1 and E_2 , respectively, such that $f(t) = g(x(t)) \cdot h(y(t))$.

It seems quite plausible that the only such functions f are exponentials; unfortunately, we were unable to decide this.

ADDED IN PROOF. A Tortrat has recently proven the conjecture stated after Theorem 4.13. The authors would also like to thank Professor Tortrat for his valuable comments.

REFERENCES

- [E] H. G. Eggleston, *Note on certain s -dimensional sets*, Fund. Math. **36** (1949), 40–43.
- [GM] C. G. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Grundlehren Math. Wiss., 238, Springer-Verlag, Berlin, Heidelberg and New York, 1979.
- [KS] J-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Actualités Sci. Indust., no. 1301, Hermann, 1963.
- [N] R. Nevanlinna, *Analytic functions*, Grundlehren Math. Wiss., 162, Springer-Verlag, Berlin, Heidelberg and New York, 1970.
- [T] A. Tortrat, *Classes de mesures singulières sur la droite, convolutions infinies de Wintner et lois indéfiniment divisibles*, J. Math. Pures Appl. **39** (1960), 231–273.

DEPARTMENT OF MATHEMATICS, WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS 02181 (Current address of N. Levenberg, Jr.)

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520 (Current address of G. J. Martin)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109 (Current address of A. L. Shields and S. Zdravkovska)