INTERIOR AND BOUNDARY REGULARITY OF SOLUTIONS TO A PLASMA TYPE EQUATION

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ABSTRACT. In this paper, we will consider a plasma type equation with homogeneous boundary condition and nonnegative initial data such that there is a finite extinction T^* . We will show that the solution is a positive classical solution in the interior of the parabolic cylinder and it decays to zero at the boundary at a certain explicit rate.

I. Introduction. In this paper, we are interested in the class of nonlinear problems

$$(1) \quad \begin{cases} \beta'(u)u_t - \Delta u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \quad u_0 \geq 0, \quad u_0 \not\equiv 0, \end{cases}$$

where β satisfies

$$(2) \quad \begin{cases} (i) & \beta \in C^2([0, \infty)), \quad \beta'(s) > 0 \text{ for } s > 0 \text{ and } \beta(0) = \beta'(0) = 0, \\ (ii) & \int_0^1 \frac{ds}{\beta^{-1}(s)} < \infty, \end{cases}$$

where β^{-1} is the inverse function of β . In this class of equations, there is the well-known plasma equation

$$(3) \quad \begin{cases} qu^{q-1}u_t - \Delta u = 0, & q > 2, \\ u|_{\partial\Omega} = 0, & u(0) = u_0 \geq 0, \quad u_0 \not\equiv 0. \end{cases}$$

We first give a brief summary of the historical background of this class of problems for which the plasma equation is a prototype.

1°. There is a finite extinction time, i.e. there exists $T^* > 0$ such that $u(T^*) \equiv 0$. This extinction property has been studied by Gregorio Diaz-Ildefonso Diaz in [3] and Crandall and Benilan in [1].

2°. As far as regularity is concerned, for $N = 1$, (1) has been studied by E. Sabanina. She proved the existence of positive classical solutions in [10]. For $N > 1$, DiBenedetto in [4] and Paul Sacks in [11] proved the existence of a bounded weak solution which is continuous in $\Omega \times (0, \infty)$. Continuity up to the boundary $\partial\Omega$ has also been studied by DiBenedetto in [4]; however no explicit estimate on the modulus of continuity has been given in any of these cases.

3°. It is a natural problem to investigate the asymptotic profile of the plasma equation as $T \rightarrow T^*$. This has been done in [2, 5].

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The aim of this paper is to investigate both the interior and boundary regularity of the solution. As far as the interior regularity is concerned, it turns out that it is intimately related to the extinction time T^* . Indeed, we will show that the solution is positive in the interior of the parabolic cylinder Q_{T^*} , i.e. we are in the nondegenerate situation in the interior of Q_{T^*} and hence we have the existence of a positive classical solution in Q_{T^*} (Sabanina proved this result for the $N = 1$ case). We will also prove that the solution decays to zero at $\partial\Omega$ at a certain explicit rate; this result is based on an explicit estimate derived in this paper. The virtue of this boundary estimate is the simplicity of the proof and the explicitness of its result.

II. Preliminaries. In this paper, we will adopt the same notations for the related functional spaces as in [8].

1°. *Weak solutions and the regularization* $\{u_n^\varepsilon\}$ of (1).

DEFINITION. Let the initial data u_0 be in $L^\infty(\Omega)$ and set $E = \{\psi \in \dot{W}_2^{1,1}(Q_T) | \psi(x, T) = 0 \text{ a.e. on } \Omega\}$. An element u of $\dot{V}_2(Q_T)$ is a weak solution of (1) if $\beta(u) \in L^2(Q_T)$ and it satisfies

$$(4) \quad \int_{Q_T} \int \beta(u) \psi_t \, dx \, dt - \int_{Q_T} \nabla u \cdot \nabla \psi \, dx \, dt + \int_{\Omega} \beta(u_0(x)) \psi(x, 0) \, dx = 0$$

for every $\psi \in E$.

It has been shown in [11] that if $u_0 \in L^\infty(\Omega)$, $u_0^n \in L_0^\infty(\Omega)$, $\|u_0^n\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$ and $u_0^n \rightarrow u_0$ in $L^p(\Omega)$ for every $1 \leq p < \infty$ as $n \rightarrow \infty$, then the standard regularization of (1), namely

$$(1, n, \varepsilon) \begin{cases} \beta'(u_n^\varepsilon) u_{n,t}^\varepsilon - \Delta u_n^\varepsilon = 0, & u_n^\varepsilon|_{\partial\Omega} = \varepsilon, \\ u_n^\varepsilon(0) = u_0^n + \varepsilon \end{cases}$$

have solutions u_n^ε which converge uniformly on compact subsets of Q_T to the weak solution $u \in C(Q_T) \cap L^\infty(Q_T)$ of (1) as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ simultaneously.

2°. *Super and subsolutions* of (1).

DEFINITION. $\beta(\bar{u}) \in C((0, T][L^1(\Omega)) \cap L^\infty(Q_T)$ is a supersolution of (1) if

$$(5) \quad \int_{\Omega} \beta(u(T)) \rho(T) \, dx \geq \int_{\Omega} \beta(u_0) \rho(0) \, dx + \int_{Q_T} (\beta(u) \rho_t + u \Delta \rho) \, dx \, dt$$

for every nonnegative $\rho \in C^{2,1}(\bar{Q}_T)$ with $\rho|_{\partial\Omega} = 0$. Subsolutions are defined analogously with $\geq$ replaced by $\leq$.

III. Main results.

THEOREM 1. Let $u(x, t)$ be the unique weak solution of (1) where $u_0 \in L^\infty(\Omega)$, $u_0 \not\equiv 0$ and $u_0 \geq 0$. Then $u(x, t)$ is a positive classical solution of (1) in the cylinder Q_{T^*} where T^* is the extinction time. More precisely, $u(x, t) > 0$ in Q_{T^*} , $u \in C^{2,1}(Q_T)$ and u satisfies $\beta'(u) u_t - \Delta u = 0$.

THEOREM 2. Let β satisfy (2) and $\partial\Omega$ be C^2 , so that it satisfies the "exterior sphere condition" (i.e. for every $x_0 \in \partial\Omega$, there exists a ball $B_R(x^*)$ of radius R (uniformly on x_0) and center x^* , with $B_R(x^*) \cap \bar{\Omega} = \{x_0\}$). Then, if u is a weak solution of (1) with $u_0 \in L^\infty(\Omega)$, for $T_0 > 0$ and $(x_0, T) \in \partial\Omega \times (0, \infty)$, $(x, T) \in \Omega \times (0, \infty)$, we have the estimate

$$u(x, T) \leq \frac{2Nk \|u_0\|_{L^\infty(\Omega)} C^{2N-1} d}{TR^{4N}} \quad \text{where } d = |x - x_0|,$$

k is a constant depending on $\beta'(\|u_0\|_{L^\infty(\Omega)})$, N , R and Ω and C satisfy $|x| < C$ for all $x \in \Omega$ (R being the radius in the exterior sphere condition).

IV. Proofs.

PROOF OF THEOREM 1. 1°. This theorem is an immediate consequence of the following lemma. Let Ω_T denote $\{(x, t) \in Q_T | t = T\}$ and let $u(x, t)$ be the bounded continuous weak solution of (1). If $u(x, t)$ vanishes at $(x^*, T^*) \in Q_T$, then it vanishes everywhere on Ω_T .

This lemma implies that u is positive in Q_{T^*} . By standard results in [8], we can obtain uniform bounds on u in $H^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ where $\alpha > 0$ and $Q_T \subset\subset Q_{T^*}$. Hence, the degeneracy has been handled and we have a positive classical solution in Q_{T^*} .

PROOF OF THE LEMMA. 2°. This proof is a generalization of Sabanina's lemma in the R^1 case [10]. For simplicity we only provide a proof for $\Omega \subseteq R^2$; the R^N case proceeds in exactly the same manner. We will argue by contradiction.

Indeed, let us assume the contrary. Then there exists a point (x^{**}, y^*, T^*) where u does not vanish and consequently we can find a region G in Q_T as shown in the figure below such that $u(x, t)$ vanishes at (x^*, y^*, T^*) but u is greater than zero on A_3 . (Here, without loss of generality, we assume $y^* = y^{**} = 0$ and $x^{**} > x^* > 0$ with $x^* = d$ and $x^{**} = R$.) We let (r, σ) denote polar coordinates and the pole of the coordinate system being located as in Figure I.

FIGURE I. The region G

We consider the following comparison function:

$$w(x, y, t) = [m(t - t^*) + c]r^\mu \cos(\pi\sigma/2\alpha) - M \quad \text{where } \mu > \pi/2\alpha$$

with $\mu > \pi/2\alpha$ and $0 < cR^\mu < M < m(T^* - t^*)d^\mu$ where m, M, c are positive constants so chosen that the inequality above holds. We define $Lu = \beta'(u)u_t - \Delta u$. Since

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \sigma^2} + \frac{1}{r} \frac{\partial w}{\partial r},$$

we have

$$\Delta \Lambda w = \Lambda \Delta w = \Lambda[m(t - t^*) + c][\mu^2 - (\pi/2\alpha)^2]r^{\mu-2} \cos(\pi\sigma/2\alpha)$$

where $\Lambda > 0$ is to be chosen later. Thus,

$$\begin{aligned} L(\Lambda w) &= \beta'(\Lambda w)w_t - \Lambda \Delta w \\ &= \beta'(\Lambda w)m\Lambda r^\mu \cos\left(\frac{\pi}{2\alpha}\sigma\right) - \Lambda[m(t - t^*) + c]\left[\mu^2 - \left(\frac{\pi}{2\alpha}\right)^2\right]r^{\mu-2} \cos\left(\frac{\pi}{2\alpha}\sigma\right) \\ &= \Lambda r^{\mu-2} \cos\left(\frac{\pi\sigma}{2\alpha}\right) \left\{ mr^2 \beta'(\Lambda w) - [m(t - t^*) + c]\left[\mu^2 - \left(\frac{\pi}{2\alpha}\right)^2\right] \right\}. \end{aligned}$$

Using the fact that $\beta'(0) = 0$ and $\beta' \in C([0, \infty))$, we can choose Λ sufficiently small so that $L(\Lambda w) \leq 0$ in G , $\Lambda w|_{A_1} = \Lambda w|_{A_2} = -\Lambda M < 0$ and, at the bottom part of G , $w = cr^\mu \cos(\pi\sigma/2\alpha) - M \leq cR^\mu - M < 0$.

Hence, by choosing Λ sufficiently small, we may obtain $(\Lambda w - u_n^\varepsilon)|_{A_3} \leq 0$. Thus in G , we have $L\Lambda w - Lu_n^\varepsilon \leq 0$ with $\Lambda w - u_n^\varepsilon \leq 0$ along A_1, A_2 and A_3 and the bottom part of G . By invoking the maximum principle for the linear case (see [9] for those standard techniques), we conclude that $\Lambda w \leq u_n^\varepsilon(x, y, t)$ in G .

In particular, $0 < \Lambda w(x^*, 0, T^*) \leq u_n^\varepsilon(x^*, 0, T^*)$. But $u_n^\varepsilon(x^*, 0, T^*) \rightarrow u(x^*, 0, T^*) = 0$ as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$. Hence, we have obtained a contradiction.

PROOF OF THEOREM 2. 1°. Again, we consider the regularization $\{u_n^\varepsilon\}$ defined by

$$(1, n, \varepsilon) \begin{cases} \beta'(u_n^\varepsilon)u_{n_t}^\varepsilon - \Delta u_n^\varepsilon = 0, & u_n^\varepsilon|_{\partial\Omega} = \varepsilon, \\ u_n^\varepsilon(0) = u_0^n + \varepsilon. \end{cases}$$

Let $U_n^\varepsilon = u_n^\varepsilon - \varepsilon$ and $\beta_\varepsilon(s) = \beta(s + \varepsilon)$; then U_n^ε satisfies

$$\begin{cases} \beta'_\varepsilon(U_n^\varepsilon)U_{n_t}^\varepsilon - \Delta U_n^\varepsilon = 0, & U_n^\varepsilon|_{\partial\Omega} = 0, \\ U_n^\varepsilon(0) = u_0^n. \end{cases}$$

Multiplying by $\rho \in C^{2,1}(\overline{Q_T})$, $\rho|_{\partial\Omega} = 0$, $\rho \geq 0$ and integrating over $\Omega \times [0, T]$, yields

$$\int_{Q_T} \beta'_\varepsilon(U_n^\varepsilon)U_{n_t}^\varepsilon \rho - \int_{Q_T} \rho \Delta U_n^\varepsilon = 0,$$

which implies that U_n^ε satisfies the corresponding integral equation

$$\int_{\Omega} \beta_\varepsilon(U_n^\varepsilon(T))\rho(T) - \int_{\Omega} \beta_\varepsilon(u_0^n)\rho(0) - \int_{Q_T} U_n^\varepsilon \Delta \rho + \beta_\varepsilon(U_n^\varepsilon)\rho_t = 0.$$

2°. Consider now any $x_0 \in \partial\Omega$, and without loss of generality, assume that the center of the exterior sphere with respect to x_0 is at $(0, \dots, 0)$. (Note that $|x| > R$ for $x \in \Omega$.) We construct the following function which turns out to a supersolution:

$$f(x, t) = \|u_0\|_{L^\infty(\Omega)}[1 - e^{-W(x)/t}]$$

where $W(x) = k[1/R^{2N} - 1/|x|^{2N}]$ and k is to be determined later. We have

$$f_t = -\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t} \frac{W(x)}{t^2}$$

and

$$\beta'_\varepsilon(f)f_t = \frac{-\beta'(f+\varepsilon)\|u_0\|_{L^\infty(\Omega)}W(x)e^{-W(x)/t}}{t^2}$$

so that

$$\beta'_\varepsilon(f)f_t \geq -\beta'(f+\varepsilon)\|u_0\|_{L^\infty(\Omega)}ke^{-W(x)/t}/R^{2N}t^2.$$

Also,

$$\nabla f = \|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}\frac{\nabla W}{t},$$

and thus,

$$\Delta f = \nabla \cdot \nabla f = -\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}\frac{|\nabla W|^2}{t^2} + \|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}\frac{\Delta W}{t}$$

where $|\nabla W(x)|^2 = 4N^2k^2/|x|^{4N+2}$. Finally,

$$-\Delta W = -\left[\frac{2N^2k}{|x|^{2N+2}} - \frac{2Nk(N+2)}{|x|^{2N+2}}\right] = \frac{4N}{|x|^{2N+2}} \geq 0.$$

Hence,

$$-\Delta f \geq \frac{4N^2k^2\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}}{|x|^{4N+2}t^2}$$

which implies that

$$\begin{aligned} \beta'_\varepsilon(f)f_t - \Delta f &= \frac{\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}}{t^2} \left[\frac{4N^2k^2}{|x|^{4N+2}} - \beta'(f+\varepsilon)W(x) \right] \\ &= \frac{\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}}{t^2} \left[\frac{4N^2k^2}{|x|^{4N+2}} - \frac{k\beta'(f+\varepsilon)}{R^{2N}} \right] \\ &= \frac{k\|u_0\|_{L^\infty(\Omega)}e^{-W(x)/t}}{t^2} \left[\frac{4N^2k}{|x|^{4N+2}} - \frac{\beta'(\|u_0\|_{L^\infty(\Omega)}(1-e^{-W(x)/t})+\varepsilon)}{R^{2N}} \right] \end{aligned}$$

and this will be ≥ 0 provided we choose k to be sufficiently large; we can do so because Ω is bounded. Here k is of order $(|x|/R)^{2N}|x|^{2N+2}$.

3°. It is now trivial to see that the f so constructed is a supersolution of U_n^ε (i.e. f satisfies the same integral equation in 1° with “=” replaced by “ $\geq$ ”). By the maximum principle (as proved and used in [5]), we have, subsolution $\leq$ supersolution. Thus, $U_n^\varepsilon(T) \leq f(T)$ so that

$$u_n^\varepsilon(x, T) \leq f(x, T) + \varepsilon = \|u_0\|_{L^\infty(\Omega)}[1 - e^{-W(x)/T}] + \varepsilon.$$

4°. Now let $(x_0, T) \in \partial\Omega \times (0, \infty)$, $x \in \Omega$ and $d = |x - x_0|$; then $u_n^\varepsilon(x, T) \leq \|u_0\|_{L^\infty(\Omega)}[1 - e^{-W(x)/T}] + \varepsilon$. For d small,

$$\begin{aligned} \|u_0\|_{L^\infty(\Omega)}[1 - e^{-W(x)/T}] + \varepsilon &\approx \|u_0\|_{L^\infty(\Omega)} \left[1 - \left(1 - \frac{W(x)}{T} \right) \right] + \varepsilon \\ &= \frac{k\|u_0\|_{L^\infty(\Omega)}}{T} \left[\frac{1}{R^{2N}} - \frac{1}{|x|^{2N}} \right] + \varepsilon \\ &= \frac{k\|u_0\|_{L^\infty(\Omega)}}{TR^{2N}|x|^{2N}} [|x|^{2N} - R^{2N}] + \varepsilon. \end{aligned}$$

But,

$$\begin{aligned} |x|^{2N} - R^{2N} &= (|x| - R)(|x|^{2N-1} + |x|^{2N-2}R + \cdots + |x|R^{2N-2} + R^{2N-1}) \\ &\leq (R + d - R)(2N)C^{2N-1} = 2NC^{2N-1}d \end{aligned}$$

where R and $|x|$ for $x \in \Omega$, are both bounded by C . Now for d small, then

$$u_n^\varepsilon(x, T) \leq \frac{\|u_0\|_{L^\infty(\Omega)} 2kNC^{2N-1}d}{TR^{4N}} + \varepsilon$$

or alternatively,

$$0 \leq u_n^\varepsilon(x, T) - \varepsilon \leq \frac{2Nk\|u_0\|_{L^\infty(\Omega)}C^{2N-1}d}{TR^{4N}},$$

where k is a constant depending on $\beta'(\|u_0\|_{L^\infty(\Omega)})$, N , R and Ω . (One can easily see the dependence by going back to the process of choosing k in step 2°.)

Recalling that u_n^ε converges uniformly on compact subsets of Q_T to u , we are done.

REMARKS. 1°. What we have obtained in the proof of Theorem 2 is not just a decay rate for $u(x, t)$ at the boundary $\partial\Omega$ but also a uniform decay rate on the regularization $\{u_n^\varepsilon\}$ at the boundary $\partial\Omega$.

2°. It should be noted that we need $\beta \in C^2([0, \infty))$ to justify our computations with respect to the u_n^ε , i.e. to invoke the maximum principle and obtain the necessary regularity on the u_n^ε . In fact, it can also be assumed that β is simply in $C'([0, \infty))$, with $\beta'(s) > 0$ for $s > 0$ and $\beta(0) = \beta'(0) = 0$; however in this case, one has to modify β by replacing it by β_ε where $\beta_\varepsilon \in C^2([0, \infty))$ and then let $\varepsilon \rightarrow 0$. In this case, although u_n^ε still converges uniformly on compact subsets of Q_T , the regularity of u will not be as good as in $C^{2,1}(Q_T)$ but somewhere in between $C(Q_T) \cap L^\infty(Q_T)$ and $C^{2,1}(Q_T)$. The optimal regularity in this case remains to be figured out.

3°. It should be noted that throughout the whole paper, we have assumed u_0 to be in $L^\infty(\Omega)$. Due to the regularizing effect, for certain range of q $u(\cdot, t)$ becomes bounded once $T > 0$ even if u_0 is implied in $L^{q-1}(\Omega)$. For this range of values of q , all the previous regularity results can be carried over for u_0 simply in $L^{q-1}(\Omega)$. For examples of this type of generalization, one may refer to the author's works in [5 and 6].

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