PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

WEAK AMENABILITY OF BANACH ALGEBRAS GENERATED BY SOME ANALYTIC SEMIGROUPS

JOŚE E. GALÉ

(Communicated by John B. Conway)

ABSTRACT. In this paper it is shown that if A is a Banach algebra generated by an analytic semigroup $(a^t)_{\mathrm{Re}\, t>0}$ such that $\|a^{1+iy}\| = O(|y|^\rho)$ $(y \in \mathbf{R})$, where $0 \leq \rho < 1/2$, then A is weakly amenable, that is, each continuous derivation from A to a commutative A-module is null.

1. Introduction. Let A be a commutative Banach algebra and X a commutative Banach A-module (this means that X is a commutative A-module with respect to the operation $(a, x) \in A \times X \mapsto a \cdot x \in X$ and there exists $k > 0$ such that $\|a \cdot x\| \leq k\|a\|\,\|x\|$ for every $a \in A$, $x \in X$), or *Banach A-module* for short. A linear mapping $D\colon A \to X$ is a *derivation* if $D(ab) = a \cdot Db + b \cdot Da$ $(a, b \in A)$. In this paper we consider only continuous derivations.

Put $H = \{t \in \mathbf{C}\colon \mathrm{Re}\, t > 0\}$. An *analytic semigroup* $(a^t)_{\mathrm{Re}\, t>0}$ in A is an analytic mapping $t \in H \mapsto a^t \in A$ such that $a^{t+s} = a^t \cdot a^s$ $(t, s, \in H)$. It is interesting to know how restrictions on the growth of $\|a^t\|$ near to the boundary of H affect the structure of A. This question can be seen detailed in [**6**, Chapter 5], where three different types of growth conditions on $\|a^t\|$ are considered. In [**4**], it is shown that if the semigroup $(a^t)_{\mathrm{Re}\, t>0}$ satisfies the condition

$$\int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy < +\infty$$

(one of those considered in [**6**]) then the Banach subalgebra B of A generated by $(a^t)_{\mathrm{Re}\, t>0}$ is regular. Further, B is also tauberian, as is noticed in [**2**]. Here, we consider a semigroup $(a^t)_{\mathrm{Re}\, t>0}$ such that $\|a^{1+iy}\| = O(|y|^\rho)$ $(|y| \to +\infty)$ for some $0 \leq \rho < 1/2$ and, under this stronger assumption, we show that B is weakly amenable, i.e. for every Banach B-module X each continuous derivation $D\colon B \to X$ is null. The proof of this fact consists in extending the derivation D to another one $\tilde{D}\colon \mathrm{Mul}(\mathscr{A}) \to Z$ where $\mathrm{Mul}(\mathscr{A})$ is the multiplier Banach algebra of a certain Banach algebra $\mathscr{A}$ and Z is a $\mathrm{Mul}(\mathscr{A})$-module. The procedure followed here to obtain $\mathscr{A}$ from B was introduced in [**3**] in relation with the problem of the existence of topologically simple Banach algebras. This method permits to construct the elements "a^{iy}" $(y \in \mathbf{R})$ as multipliers of $\mathscr{A}$ to conclude, by application of a result of [**1**], that $D = 0$.

Now we recall standard notions and facts. Let A be a commutative Banach algebra and let X be a Banach A-module. Put $A^\perp = \{b \in A\colon ab = 0$ for all

Received by the editors June 1, 1987 and, in revised form, October 9, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46J35.
Key words and phrases. Banach algebra, Weak amenability, analytic semigroup.
This research has been supported by the Spanish CAICYT Grant number 0804-84.

$a \in A\}$ and $X^\perp = \{x \in X : ax = 0$ for all $a \in A\}$. The quotient Banach space $X/X^\perp$ is an $A/A^\perp$-module with respect to the operation $\bar{a} \cdot \bar{x} = \overline{a \cdot x}$ ($a \in A$, $x \in X$, $\bar{a} = a + A^\perp$, $\bar{x} = x + X^\perp$) since if $a_1 - a_2 \in A^\perp$ and $x_1 - x_2 \in X^\perp$ then $(a_1x_1 - a_2x_2)b = (a_1 - a_2)bx_1 + a_2(x_1 - x_2)b = 0$ for all $b \in A$. Further, $X/X^\perp$ is a Banach $A/A^\perp$-module because

$$\|\overline{a \cdot x}\| \leq \|a \cdot x\| = \inf_{z \in X^\perp} \|a \cdot x - a \cdot z\| \leq \|a\| \, \|\bar{x}\|$$

so if $r \in A^\perp$, $\|\overline{ax}\| = \|\overline{(a - r) \cdot x}\| \leq \|a - r\| \, \|\bar{x}\|$ whence $\|\overline{a \cdot x}\| \leq \|\bar{a}\| \, \|\bar{x}\|$. Suppose now that $[AA]^- = A$. Let $D\colon A \to X$ be a continuous derivation and define $D^\perp\colon A/A^\perp \to X/X^\perp$ by $D^\perp(\bar{a}) = \overline{D(a)}$ ($a \in A$). $D^\perp$ is well defined: if $r \in A^\perp$ and $a, b \in A$ then $D(r) \cdot ab = -r \cdot D(ab) = -r(a \cdot Db + b \cdot Da) = 0$. Moreover, $D^\perp$ is a derivation and

$$\|D^\perp(\bar{a})\| = \inf_{r \in A^\perp} \|D^\perp(\overline{a - r})\| = \inf_{r \in A^\perp} \|\overline{D(a - r)}\| \leq \|D\| \inf_{r \in A^\perp} \|a - r\| = \|D\| \, \|\bar{a}\|$$

which implies that $D^\perp$ is continuous.

Let $\mathscr{A}$ be a commutative Banach algebra and $(e_n)_{n \geq 1}$ a *bounded approximate identity* in $\mathscr{A}$, i.e. $\|e_n\| \leq k'$ ($n \geq 1$) for some $k' \geq 1$ and $ae_n \underset{n}{\to} a$ for every $a \in \mathscr{A}$. For a Banach $\mathscr{A}$-module X we denote by $\mathrm{Hom}(\mathscr{A}; X)$ the set of mappings $T\colon \mathscr{A} \to X$ such that $T(ab) = a \cdot Tb$ ($a, b \in \mathscr{A}$). In fact, each T is linear and bounded [**5**] and $\mathrm{Hom}(\mathscr{A}; X)$ is a Banach space under the usual operations and norm $\|T\| := \sup_{\|a\| \leq 1} \|Ta\|$. Assume $\mathscr{A} \cdot X = X$. In that case, if $x \in X$ and $e_n x = 0$ ($n \geq 1$) then $x = 0$ and the mapping $x \in X \mapsto T_x \in \mathrm{Hom}(\mathscr{A}, X)$ where $T_x(a) = a \cdot x$ ($a \in \mathscr{A}$), is an injection. Also, for $\mathscr{A} \cdot X = X$, X is a Banach $\mathrm{Mul}(\mathscr{A})$-module with respect to the operation $(T, x) \in \mathrm{Mul}(\mathscr{A}) \times X \mapsto T \cdot x = Ta \cdot y \in X$, where $a \in \mathscr{A}$, $y \in X$ and $x = a \cdot y$ (see [**3**, p. 96]). Note that $\mathrm{Hom}(\mathscr{A}; \mathscr{A}) = \mathrm{Mul}(\mathscr{A})$.

PROPOSITION (1.1). *Let $\mathscr{A}$ be as above, let Y be a Banach $\mathscr{A}$-module and let $D\colon \mathscr{A} \to Y$ be a derivation. Then there exists a (unique) derivation $\tilde{D}\colon \mathrm{Mul}(\mathscr{A}) \to \mathrm{Hom}(\mathscr{A}, Y)$ such that $\tilde{D}(a) = D(a)$ for every $a \in \mathscr{A}$.*

PROOF. Since $\mathscr{A}$ possesses a bounded approximate identity we have that $\mathscr{A} \cdot \mathscr{A} = \mathscr{A}$ and $\mathscr{A} \cdot Y$ is a closed subspace of Y, by Cohen's Theorem [**6**]. Put $X = \mathscr{A}Y$. Then $\mathscr{A}X = \mathscr{A}\mathscr{A}Y = \mathscr{A}Y = X$, $D(\mathscr{A}) = D(\mathscr{A}\mathscr{A}) \subset \mathscr{A}Y + \mathscr{A}Y = X$ and we can define $\tilde{D}\colon \mathrm{Mul}(\mathscr{A}) \to \mathrm{Hom}(\mathscr{A}, Y)$ as $\tilde{D}(T)(a) = D(Ta) - T \cdot Da$ ($a \in \mathscr{A}$, $T \in \mathrm{Mul}(\mathscr{A})$). It is easy to verify that $\tilde{D}$ is a derivation such that $\tilde{D}(a) = D(a)$ for every $a \in \mathscr{A}$.

2. Semigroups and weak amenability. Let A be a Banach algebra and let $(a^t)_{\mathrm{Re}\, t > 0}$ be an analytic semigroup in A. It is well known that $[a^tA]^- = [a^sA]^-$ ($t, s \in H$) [**6**, p. 7]. Throughout this section we suppose that A is generated by $(a^t)_{\mathrm{Re}\, t > 0}$. A consequence of this, when $(a^t)_{\mathrm{Re}\, t > 0}$ is such that $\|a^{1+iy}\| = O(|y|^\rho)$ ($|y| \to +\infty$) for some $0 \leq \rho < 1/2$, is that A is also generated by the subalgebra of all polynomials in a^1, say $P(a)$ [**4**, p. 379]. Then $P(a) \subset a^{1/2}A \subset A$ and so $[a^tA]^- = [a^{1/2}A]^- = P(a)^- = A$ ($t \in H$). Without loss of generality we assume $\|a^t\| \leq 1$ for every $t \geq 1$. Put $a = a^1$.

LEMMA (2.1). *Let A be a Banach algebra generated by an analytic semigroup $(a^t)_{\operatorname{Re} t>0}$ and such that $A^\perp = (0)$. Let $\omega\colon \mathbf{R} \to \mathbf{R}$ such that $\omega(y) > 0$, $\omega(y_1+y_2) \leq \omega(y_1)\cdot\omega(y_2)$ and $\|a^{1+iy}\| \leq \omega(y)$ $(y, y_1, y_2 \in \mathbf{R})$. Then there exists a continuous injection with dense range $A \hookrightarrow \mathscr{A}$ where $\mathscr{A}$ is a Banach algebra with a bounded approximate identity such that $a^{iy}\alpha := \lim_{t\to iy} a^t\alpha$ exists in $\mathscr{A}$ for every $\alpha \in \mathscr{A}$ $(y \in \mathbf{R})$. Moreover $(a^{iy})_{y\in R} \subseteq \operatorname{Mul}(\mathscr{A})$ and $\|a^{iy}\| \leq \omega(y)$ $(y \in \mathbf{R})$.*

PROOF. Put $h(t) = \omega(\operatorname{Im} t)^{-1}$ $(\operatorname{Re} t > 0)$. First, note that, by considering $b^t = a^{2t}$ $(\operatorname{Re} t > 0)$ and $\bar\omega(y) = \omega(2y)$ $(y \in \mathbf{R})$, we can assume that $\sup_{\operatorname{Re} t\geq 0} \|h(t)a^{1+t}\| < +\infty$ since

$$\begin{aligned}\|b^{1+t}\| = \|a^{2+2t}\| &\leq \|a^{1+i2\operatorname{Im} t}\|\,\|a^{1+\operatorname{Re} t}\| \\ &\leq \omega(2\operatorname{Im} t) = \bar\omega(\operatorname{Im} t) \qquad (\operatorname{Re} t \geq 0).\end{aligned}$$

Now, for $y \in \mathbf{R}$, put $\mathscr{D}_y = \{b \in A\colon a^{1+iy}b = ac \text{ for some } c \in A\}$. If $b \in \mathscr{D}_y$ then c is unique because $A^\perp = (0)$. Define $T_y(b) := c (b \in \mathscr{D}_y)$. Then T_y is a densely defined operator on A since $a \in \mathscr{D}_y$. Moreover, it is routine to verify that T_y is a closed operator. We will write a^{iy} instead of T_y. Observe that a^0 is the identity.

Consider

$$I = \left\{ u \in A\colon u \in \mathscr{D}_y \ (y \in \mathbf{R}), \sup_{\operatorname{Re} t\geq 0, |\lambda_t|\leq h(t)} \|\lambda_t a^t u\| < +\infty \right\}.$$

I is a nontrivial dense ideal of A because $a \in I$. Put

$$p(u) = \sup_{\operatorname{Re} t\geq 0, |\lambda_t|\leq h(t)} \|\lambda_t a^t u\| \qquad (u \in I).$$

The mapping p is a complete norm on I such that $\|u\| \leq p(u)$, $p(bu) \leq \|b\|p(u)$ $(b \in A,\ u \in I)$. For $b \in A$, define $q(b) = \sup_{u\in I, p(u)\leq 1} p(bu)$. Then q is an algebra norm on A such that $q(b) \leq \|b\|$ $(b \in A)$. Let $\mathscr{A}$ be the q-completion of A. Then $\mathscr{A}$ is generated by $(a^t)_{\operatorname{Re} t>0}$, and

$$q(h(t)a^t) = \sup_{\operatorname{Re} s\geq 0, p(u)\leq 1} \|\lambda_s h(t) a^{s+t} u\| \leq p(u) \leq 1 \qquad (\operatorname{Re} t > 0).$$

In particular $q(a^t) \leq \omega(0)$ for every $t > 0$ and $(a^{1/n})_{n\geq 1}$ is a bounded approximate identity for $\mathscr{A}$ because $a^{1+1/n} \underset{n}{\to} a$. Finally, if $y \in \mathbf{R}$ and $\alpha \in \mathscr{A}$, $\lim_{t\to iy} a^t\alpha$ exists (see [**6**, p. 84]) and we have

$$\|a^{iy}\| = \sup_n \|a^{iy}(a^{1/n})\| = \sup_n \|a^{1/n+iy}\| \leq \omega(y). \quad \square$$

The basic idea in the proof of Lemma (2.1) to obtain $\mathscr{A}$ from A by means of I is contained in [**3**, §7]. Furthermore, according to the terminology of [**3**, Lemma (2.1)] is really a consequence of the fact that $(\lambda_t a^t)_{\operatorname{Re} t\geq 0, |\lambda_t|\leq h(t)}$ is a *stable under products from a pseudobounded set of regular quasimultipliers* of A (see §2 of [**3**] for definitions). We give here this detailed proof of Lemma (2.1) for the sake of completeness.

LEMMA (2.2). *Let $A, \mathscr{A}$ be as in Lemma (2.1) and let X be a Banach A-module such that $X^\perp = (0)$. If $D\colon A \to X$ is a continuous derivation then there exist a*

Banach $\mathscr{A}$-module Z which contains continuously X and a continuous derivation $\tilde{D}\colon \mathscr{A} \to Z$ such that the restriction of $\tilde{D}$ to A is D.

PROOF. We will use the notations of the proof of Lemma (2.1). The number

$$|||x||| = \sup_{\operatorname{Re} t\geq 0, |\lambda_t|\leq h(t), p(u)\leq 1} \|\lambda_t a^t ux\| \qquad (x \in X)$$

defines a norm on X such that $|||x||| \leq \|x\|$. Let Y be the completion of X with respect to $|||\ |||$. Since $|||b \cdot x||| \leq q(b)\|x\|$ $(b \in A,\ x \in X)$ we can define by continuity a continuous bilinear mapping $(\alpha, x) \in \mathscr{A} \times X \mapsto a \cdot x \in Y$ which extends the A-module operation on X. Observe also that $|||b \cdot x||| \leq \|b\|\ |||x|||$ $(b \in A,\ x \in X)$ whence, defining again by continuity $b \cdot x'$ $(x' \in Y)$, Y becomes a Banach A-module. Let Z be the vector space of all linear operators $T\colon I \to Y$ such that $T(bu) = b \cdot Tu$ $(b \in A,\ u \in I)$ and $\sup_{p(u)\leq 1} |||Tu||| < +\infty$. Under the norm $|||T||| = \sup_{p(u)\leq 1} |||Tu|||$, Z is a Banach space and the mapping $x \in X \mapsto T_x \in X$, where $T_x(u) = u \cdot x$, $(u \in I)$ is injective and continuous. If we define $(b \cdot T)(u) := T(b \cdot u)$ for every $b \in A$, $T \in Z$, $u \in I$, then $b \cdot T \in Z$ and

$$|||b \cdot T||| = \sup_{p(u)\leq 1} |||T(b \cdot u)||| \leq |||T|||q(b),$$

so Z admits a structure of Banach $\mathscr{A}$-module. Now, we have

$$\begin{aligned} |||Db \cdot u||| &= |||D(b \cdot u) - b \cdot Du||| \leq |||D(b \cdot u)||| + |||b \cdot Du||| \\ &\leq \|D(b \cdot u)\| + q(b)\|Du\| \\ &= \|D\|p(bu) + q(b)\|D\|p(u) \\ &\leq 2\|D\|q(b)p(u) \qquad (u \in I,\ b \in A) \end{aligned}$$

Therefore $|||Db||| \leq 2\|D\|q(b)$ $(b \in A)$ so we define $\tilde{D}$ as the extension (by continuity) of D on $\mathscr{A}$. It is easy to check that $\tilde{D}$ is a continuous derivation. □

THEOREM (2.3). *Let A be a Banach algebra generated by an analytic semigroup $(a^t)_{\operatorname{Re} t>0}$ such that $\|a^{1+iy}\| = O(|y|^\rho)$ $(|y| \to +\infty)$ for some $0 \leq \rho < 1/2$. Then A is weakly amenable.*

PROOF. Consider a Banach A-module X and a continuous derivation $D\colon A \to X$. First, we suppose that $A^\perp = (0)$ and $X^\perp = (0)$. We have that $\|a^{1+iy}\| \leq k(1 + |y|)$ $(y \in \mathbf{R})$ for some $k \geq 1$. Therefore if we put $\omega(y) = k(1 + |y|)^\rho$ $(y \in \mathbf{R})$, Lemmas (2.1) and (2.2) apply. Let $\mathscr{A}, Z, \tilde{D}$ be as in these lemmas. By Proposition (1.1) there exists a continuous derivation $\Delta\colon \operatorname{Mul}(\mathscr{A}) \to \operatorname{Hom}(\mathscr{A}; Z)$ which extends $\tilde{D}$. Take $y \in \mathbf{R}$. Then, by Lemma (2.1), $\|a^{iny}\|\ \|a^{-iny}\|/n \to 0$ as $n \to \infty$. Therefore, by Theorem 1.4, 2B of [**1**] we deduce that $\Delta(a^{iy}) = 0$. Let l be a continuous linear form on $\operatorname{Hom}(\mathscr{A}; Z)$. The mapping $l(\Delta(a^t)a^2)$ is analytic and bounded on H, and

$$\begin{aligned} l(\Delta(a^t)a^2) &= l(\Delta(a^{1+t}) \cdot a - a^{1+t} \cdot \mathbf{\Delta} a) \xrightarrow[t\to iy]{} l(\Delta(a^{1+iy})a - a^{1+iy} \cdot a) \\ &= l(\Delta(a^{iy}) \cdot a^2) = 0 \qquad \text{for every } y \in \mathbf{R}. \end{aligned}$$

It follows that $D(a^t)a^2 = \Delta(a^t) \cdot a^2 = 0$ so $D(a) \cdot a^2 = 0$. Then $D(a) \cdot a \in X^\perp = (0)$, i.e. $D(a) \in X^\perp = (0)$.

Secondly, in the general case we consider $D^{\perp}\colon A/A^{\perp} \to X/X^{\perp}$ as in the Introduction. According to the first part of this proof, we obtain that $\overline{Da} = D^{\perp}(\bar{a}) = \bar{0}$, i.e. $Da \in X^{\perp}$. Therefore $a \cdot Da = 0$ and so $D(a^n) = 0$ for every $n \geq 2$. But this implies that $Da^t = 0$ $(t \in H)$ (see [**4**, p. 379], again). Consequently $Da = 0$ and $D = 0$. □

REMARK. The condition imposed on ρ in the preceding theorem cannot be improved. To see this, consider the Beurling algebra

$$L^1(R, \omega_\rho) := \left\{ f\colon \mathbf{R} \to \mathbf{C} \text{ measurable: } \|f\|_\rho := \int_{-\infty}^{+\infty} |f(u)|\omega_\rho(u)\, du < +\infty \right\}$$

where $\rho > 0$ and $\omega_\rho(u) = (1+|u|)^\rho$ $(u \in \mathbf{R})$ and denote by A_ρ the Banach subalgebra of $L^1(R, \omega_\rho)$ generated by the semigroup

$$a^t(u) = \frac{1}{\sqrt{\pi t}} e^{-u^2/t} \qquad (u \in \mathbf{R},\ \operatorname{Re} t > 0),$$

which is analytic in $L^1(R, \omega_\rho)$. Then, if $y \in \mathbf{R}$,

$$\|a^{1+iy}\|_\rho = \frac{1}{\sqrt{\pi(1+y^2)}} \int_{-\infty}^{+\infty} e^{-u^2/1+y^2}(1+|u|)^\rho\, du$$

and, setting $v = u/\sqrt{1+y^2}$,

$$\|a^{1+iy}\|_\rho \approx \int_{-\infty}^{+\infty} e^{-v^2}|v|^\rho |y|^\rho\, dv = O(|y|^\rho).$$

Now, for $\rho \geq 1/2$, put

$$X_\rho := \left\{ f\colon \mathbf{R} \to \mathbf{C} \text{ measurable: } \|f\| = \int_{-\infty}^{+\infty} \frac{|f(u)|}{1+|u|}\omega_\rho(u)\, du < +\infty \right\}.$$

X_ρ is a Banach space. Moreover, if $f \in X_\rho$ and $g \in A_\rho$ then $f * g \in X_\rho$ and $\|f * g\| \leq \|f\|\,\|g\|_\rho$. Thus X_ρ is a Banach A_ρ-module. Define $(Dg)(u) = ug(u)$ $(g \in A_\rho,\ u \in \mathbf{R})$. Then $D\colon A_\rho \to X_\rho$ is a nonzero continuous derivation. Thus A_ρ is not weakly amenable. (This is a copy of the proof of Theorem (2.3) in [**1**].)

ACKNOWLEDGEMENTS. I would like to thank Professor H. G. Dales for his useful comments about the original manuscript of this paper.

REFERENCES

1. W. G. Badé, P. C. Curtis and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, preprint.
2. J. C. Candeal, *Sobre condiciones de suficiencia para la propiedad tauberiana de Wiener*, Actas del VII Congreso de Matemáticos de Expresión Latina, Coimbra, Portugal, 1985.
3. J. Esterle, *Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras*, Lecture Notes in Math., vol. 975, Springer-Verlag, Berlin and New York, 1983, pp. 66–162.
4. J. Esterle and J. E. Galé, *Regularity of Banach algebras generated by analytic semigroups satisfying some growth conditions*, Proc. Amer. Math. Soc. **92** (1984), 377–380.
5. B. E. Johnson, *Continuity of centralizers on Banach algebras*, J. London Math. Soc. **41** (1966), 639–640.
6. A. M. Sinclair, *Continuous semigroups in Banach algebras*, London Math. Soc. Lecture Notes, vol. 63, Cambridge Univ. Press, 1982.

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, 50009-ZARAGOZA, SPAIN