A WEAK KRASNOSEL'SKII THEOREM IN R^d

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ABSTRACT. Let S be a compact, locally starshaped set in R^d , and let k be a fixed integer, $0 \leq k \leq d$. If every $d - k + 1$ points of S are clearly visible via S from a common point, then for every k -flat F' there exists a translate F of F' such that the following holds:

To each point s_0 in $S \sim F$ there correspond a point s_m in F and a polygonal path $\bigcup\{[s_{i-1}, s_i]: 1 \leq i \leq m\}$ in $S \cap \text{aff}(s_0 \cup F)$ with $\text{dist}(s_i, F) < \text{dist}(s_{i-1}, F)$, $1 \leq i \leq m$.

If $k = 0$ or $k = d - 1$, then each point of S sees via S some point of F . Moreover, if $k = 1$, then F can be chosen so that $F \cap S$ is convex.

Introduction. We begin with some preliminary definitions. Let S be a set in R^d . For points x, y in S , we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S . Similarly, x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees via S each point of $N \cap S$. For $\delta > 0$, x is δ -*visible* from y via S if and only if y sees via S each point of $N(x, \delta) \cap S$, where $N(x, \delta)$ is the neighborhood of x having radius δ . Set S is *starshaped* if and only if there is some point p of S such that p sees each point of S via S , and the set of all such points p is called the (convex) *kernel* of S , denoted $\ker S$. Finally, set S is *locally starshaped* at point s if and only if for some neighborhood N of s , $N \cap S$ is starshaped with $p \in \ker(N \cap S)$.

A well-known theorem of Krasnosel'skii [9] states that if S is a nonempty compact set in R^d , S is starshaped if and only if every $d + 1$ points of S are visible via S from a common point. The proof is based on the following theorem of Helly [4]: If $\mathcal{F}$ is a family of compact sets in R^d , $\bigcap\{F: F \text{ in } \mathcal{F}\} \neq \emptyset$ if and only if every $d + 1$ members of $\mathcal{F}$ have a nonempty intersection. Various analogues of Helly's theorem have been found when the number $d + 1$ is replaced by a smaller number. (See Horn and Valentine [6], Horn [5], Klee [7].) In particular, Horn [5] has proved that for $1 \leq k \leq d$, if every $d - k + 1$ members of $\mathcal{F}$ have a nonempty intersection, then for any $(k - 1)$ -flat F_1 , there corresponds a k -flat F_2 containing F_1 and meeting every member of $\mathcal{F}$. Klee [8] has pointed out that it would be interesting to investigate corresponding weak analogues of Krasnosel'skii's theorem as well.

In [1], Horn's theorem and the notion of clearly visible were used to obtain the following result: Let S be a nonempty compact connected set in R^2 , and assume that every two points of local nonconvexity of S are clearly visible via S from a common point. Then for any point y in R^2 , there is a line L through y such that $L \cap S$ is convex and every point of S sees via S a point of L . Here a similar result is obtained in higher dimensions: Let S be a compact, locally starshaped set in R^d ,

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and let k be a fixed integer, $0 \leq k \leq d$. If every $d - k + 1$ points of S are clearly visible via S from a common point, then for every k -flat F' there exists a translate F of F' such that each point of S is joined to F by a certain kind of polygonal path in S . Moreover, if $k = 1$, then F can be chosen so that $F \cap S$ is convex.

The following familiar terminology will be used throughout the paper: $\text{conv } S$, $\text{aff } S$, and $\text{cl } S$ will denote the convex hull, affine hull, and closure, respectively, for set S . The distance from point x to set S will be denoted $\text{dist}(x, S)$. For distinct points x and y , $R(x, y)$ will be the ray from x emanating through y , and $\text{cone}(x, S)$ will represent $\bigcup \{R(x, s) : s \in S, x \neq s\}$. Finally, for $s \in S$, we define set

$$A_s = \{y : s \text{ is clearly visible from } y \text{ via } S\}.$$

The reader is referred to Valentine [11] and to Lay [10] for a discussion of these concepts.

The results. We begin with three easy lemmas.

LEMMA 1. *Let $\mathcal{C}$ be a finite family of convex sets in R^d , and let k be a fixed integer, $0 \leq k \leq d$. If every $d - k + 1$ members of $\mathcal{C}$ have a nonempty intersection, then for every k -flat F' , there is a translate F of F' which intersects every member of $\mathcal{C}$.*

PROOF. For convenience of notation, assume that F' is the k -flat

$$\{(x_1, \dots, x_k, 0, \dots, 0) : x_i \text{ real}, 1 \leq i \leq k\},$$

and let H denote its orthogonal complement

$$\{(0, \dots, 0, x_{k+1}, \dots, x_d) : x_i \text{ real}, k+1 \leq i \leq d\}.$$

For each C in $\mathcal{C}$, project C onto H in the direction of F to obtain convex set C' . Using our hypothesis, every $d - k + 1$ of the C' sets meet in the $(d - k)$ -flat H . Hence by Helly's theorem, $\bigcap \{C' : C \text{ in } \mathcal{C}\} \neq \emptyset$. Select point p in this intersection. Then the translate of F' containing p necessarily intersects every member of $\mathcal{C}$ and satisfies the lemma.

LEMMA 2. *Let S be a set in R^d , $q \in S$. If $x \in \text{conv } A_q$, then there is a neighborhood N of q such that for every v in $N \cap S$, $x \in \text{conv } A_v$.*

PROOF. Using Carathéodory's theorem, select points $x_1, \dots, x_k$ in A_q with $x \in \text{conv}\{x_1, \dots, x_k\}$ and $k \leq d + 1$. By the definition of clearly visible, choose a convex neighborhood N of q so that each point x_i sees via S every point of $N \cap S$, $1 \leq i \leq k$. Then for v in $N \cap S$, v is clearly visible via S from each x_i . That is, $\{x_1, \dots, x_k\} \subset A_v$, and $x \in \text{conv}\{x_1, \dots, x_k\} \subset \text{conv } A_v$.

LEMMA 3. *Let S be a set in R^d , $v \in S$. If $x \in \text{conv } A_v$ and $x \neq v$, then there is an x' , $v < x' \leq x$, with $[v, x'] \subseteq S$.*

PROOF. This follows from the argument in [3, Lemma 2].

THEOREM 1. *Let S be a compact, locally starshaped set in R^d , and let k be a fixed integer, $0 \leq k \leq d$. If every $d - k + 1$ points of S are clearly visible via S from a common point, then for every k -flat F' , there exists a translate F of F' such that the following holds:*

To each point s_0 of $S \sim F$ there correspond a point s_m in F and a polygonal path $\bigcup\{[s_{i-1}, s_i]: 1 \leq i \leq m\}$ in $S \cap \text{aff}(s_0 \cup F)$ with $\text{dist}(s_i, F) < \text{dist}(s_{i-1}, F)$, $1 \leq i \leq m$.

If $k = 0$ or $k = d - 1$, then each point of S sees via S some point of F . Moreover, if $k = 1$, then F can be chosen so that $F \cap S$ is convex.

PROOF. Using [2, Lemma 1], select a $\delta > 0$ such that every $d - k + 1$ points of S are δ -visible via S from a common point. The collection of δ -neighborhoods $\{N(s, \delta): s \in S\}$ covers the compact set S , so we may reduce to a finite subcover, say $\{N(s_i, \delta): 1 \leq i \leq j\}$.

For every s in S , define set $T_s = \{y: s \text{ is } \delta\text{-visible from } y \text{ via } S\}$. By our choice of δ , every $d - k + 1$ points of S are δ -visible via S from a common point. Hence every $d - k + 1$ of the convex sets in $\{\text{conv } T_{s_i}: 1 \leq i \leq j\}$ have a nonempty intersection. By Lemma 1, there is a translate F of F' which intersects each set $\text{conv } T_{s_i}$, $1 \leq i \leq j$.

Moreover, we assert that F meets every set $\text{conv } A_s$ as well: For $s \in S$, select a corresponding point s_i , $1 \leq i \leq j$, with $s \in N(s_i, \delta)$, and choose $w \in F \cap \text{conv } T_{s_i} \neq \emptyset$. Using Carathéodory's theorem, there exist points $z_{i,1}, \dots, z_{i,d+1}$ in T_{s_i} such that $w \in \text{conv}\{z_{i,1}, \dots, z_{i,d+1}\}$. Certainly each point $z_{i,r}$ sees via S each point of $N(s_i, S) \cap S$, so s is clearly visible via S from each $z_{i,r}$, $1 \leq r \leq d + 1$. Thus $\{z_{i,1}, \dots, z_{i,d+1}\} \subset A_s$, and $w \in \text{conv}\{z_{i,1}, \dots, z_{i,d+1}\} \cap F \subset (\text{conv } A_s) \cap F \neq \emptyset$. The assertion is established.

We will show that for each point s_0 in $S \sim F$ there is an appropriate polygonal path from s_0 to some point of F . Select a point x in $(\text{conv } A_{s_0}) \cap F \neq \emptyset$. Then $x \neq s_0$, and using Lemma 3, there exists a point x' , $s_0 < x' \leq x$, with $[s_0, x'] \subseteq S$. Observe that $[s_0, x'] \subseteq \text{aff}(s_0 \cup F)$ and $\text{dist}(x', F) < \text{dist}(s_0, F)$. Since S is compact, of all points y with $[s_0, y] \subseteq S \cap \text{aff}(s_0 \cup F)$, there is a point s_1 (not necessarily unique) with $\text{dist}(s_1, F)$ minimal. Of course $\text{dist}(s_1, F) < \text{dist}(s_0, F)$. If $s_1 \in F$, the argument is finished; otherwise, repeat the procedure for s_1 . By an obvious induction, if $s_n \notin F$ for some $1 \leq n$, we obtain a polygonal path

$$\bigcup\{[s_{i-1}, s_i]: 1 \leq i \leq n + 1\}$$

in $S \cap \text{aff}(s_0 \cup F)$ with $\text{dist}(s_i, F) < \text{dist}(s_{i-1}, F)$, $1 \leq i \leq n + 1$.

It remains to show that for some m , $s_m \in F$. Otherwise, by our induction we will define an infinite sequence $\{s_i: 1 \leq i\}$ in $S \cap \text{aff}(s_0 \cup F)$. Pass to a converging subsequence $\{t_i: 1 \leq i\}$, and assume that $\{t_i: 1 \leq i\}$ converges to t in $S \cap \text{aff}(s_0 \cup F)$. Since S is locally starshaped, there is an integer l such that $[t, t_i] \subseteq S$ for $i \geq l$. Letting s'_{l+1} denote the successor of t_l in our induction, we have $\text{dist}(t, F) \leq \text{dist}(t_{l+2}, F) < \text{dist}(t_{l+1}, F) \leq \text{dist}(s'_{l+1}, F)$. That is, $[t, t_l] \subseteq S \cap \text{aff}(s_0 \cup F)$ and t is closer to F than s'_{l+1} is to F . However, this contradicts our choice of s'_{l+1} as a closest point to F seen by t_l in $S \cap \text{aff}(s_0 \cup F)$. We have a contradiction, and for some m , $s_m \in F$. Then the polygonal path $\bigcup\{[s_{i-1}, s_i]: 1 \leq i \leq m\}$ satisfies the theorem.

Finally, we examine various special values for k . If $k = 0$, then S is starshaped by Krasnosel'skii's theorem, so the result is trivial. If $k = d - 1$, then the flats F' and F in the theorem are hyperplanes. Recall that by the proof above, for every $s \in S$, $F \cap (\text{conv } A_s) \neq \emptyset$. Hence set A_s cannot lie entirely in either of the open halfspaces F_1 or F_2 determined by F . If we select points $w_i \in (\text{cl } F_i) \cap A_s \neq \emptyset$, $i = 1, 2$, one of $[w_1, s]$ or $[w_2, s]$ necessarily meets F at some point p . Then s sees

FIGURE 1

p via S , and p satisfies the theorem. (Moreover, minor modifications in the proof show that the case for $k = d - 1$ holds with clearly visible replaced by visible and without the requirement that S be locally starshaped.)

We conclude with the case for $k = 1$. That is, if $k = 1$, then for F produced in the argument above, $F \cap S$ is convex. Suppose on the contrary that there exist points a, b in $F \cap S$ with $[a, b] \not\subseteq S$. Without loss of generality, assume that $(a, b) \subseteq \sim S$. By our hypothesis, points a and b are (clearly) visible via S from a common point s . Since $(a, b) \not\subseteq S$, $s \notin F$, and points a, b, s determine a plane $\pi = \text{aff}(s \cup F)$. Let K be the component of $\text{conv}\{a, b, s\} \cap S$ containing (a, b) , and let $M \neq F$ be a line parallel to F and supporting $\text{conv}(\text{cl } K)$. Moreover, let M_1 and M_2 denote the halfplanes of π determined by M , labeled so that $F \subseteq M_2$. Then $K \subseteq M_2$, also. We will show that for some boundary point q of S on M , $(\text{conv } A_q) \cap \pi \subseteq \text{cl } M_1$, thereby contradicting the fact that F meets $\text{conv } A_q$.

Let q be an extreme point of $\text{conv}(\text{cl } K)$ in M . Since $\sim S$ is open, $q \notin \sim S$, and it is easy to see that $q \in (\text{bdry } S) \cap (\text{bdry } K)$. Let $x \in (\text{conv } A_q) \cap \pi \neq \emptyset$, and select a neighborhood N of q which satisfies Lemma 2. That is, for $v \in N \cap S$, $x \in \text{conv } A_v$.

We will show that $x \in \text{cl } M_1$. Assume that $x \in M_2$ to obtain a contradiction. Then $\text{cone}(x, M \cap N)$ contains neighborhoods of q , and since $q \in \text{bdry } K$ and $K \subseteq M_2$, it follows that $\text{conv}(x \cup (M \cap N))$ contains some point $w \in N \cap K \subseteq M_2$. (See Figure 1). The ray $R(x, w)$ meets $M \cap N$ at some point z . Since M supports $\text{conv}(\text{cl } K)$, clearly $z \notin K$ and hence $(w, z]$ meets $\text{bdry } K$ at some point v , where v may be chosen closest to w . Then $[w, v] \subseteq K$ and $v \in N \cap \text{bdry } K \subseteq N \cap S$. Since N satisfies Lemma 2 and $v \in N \cap S$, it follows that $x \in \text{conv } A_v$. Moreover, by Lemma 3, there is an $x', v < x' \leq x$, with $[v, x'] \subseteq S$. However, this is impossible since $(v, w] \subseteq K \subseteq \sim S$. Our assumption is false, $x \notin M_2$, and $x \in \text{cl } M_1$.

We have proved that $(\text{conv } A_q) \cap \pi \subseteq \text{cl } M_1$. However, $F \subseteq M_2$ so this contradicts the fact that F meets $\text{conv } A_q$. Our original supposition must be false, and $F \cap S$ is indeed convex, finishing the proof of the theorem.

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