A CHARACTERIZATION OF THE GENERALIZED VERONESE SURFACES

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ABSTRACT. We prove that compact m -regular minimal surfaces in a sphere are generalized Veronese surfaces if the Gaussian curvature satisfies an inequality.

The Gaussian curvature of a surface is an intrinsic value. It is well known that minimal surfaces of constant curvature in a sphere are rigid [1]. These surfaces are called *generalized Veronese surfaces* in [3 or 5]. Now, we may conjecture that generalized Veronese surfaces can be characterized only by the Gaussian curvature.

In this paper, we will prove that this conjecture is true for m -regular minimal surfaces (m -regular means that the k th normal space N_k satisfies $\dim N_k > \dim N_{k-1}$ for $k = 1, 2, \dots, m$, where $\dim N_0 = 0$, and 0-regular mapping means an ordinary regular one). That is, we prove the following:

THEOREM. *Let M be a compact connected oriented surface minimally immersed in a unit sphere through the m -regular immersion. If its Gaussian curvature K satisfies*

$$\frac{2}{(m+2)(m+3)} \leq K \leq \frac{2}{(m+1)(m+2)}, \quad 0 \leq m,$$

then M is a generalized Veronese surface.

1. Preliminaries. Let $\tilde{M}$ be a $(2+\nu)$ -dimensional Riemannian manifold of constant curvature $\tilde{c}$, and M a 2-dimensional Riemannian manifold which is immersed in $\tilde{M}$ by the immersion $x : M \rightarrow \tilde{M}$. Let $(e_1, e_2, \dots, e_{2+\nu})$ be an orthonormal frame field over $\tilde{M}$ such that (e_1, e_2) is an orthonormal frame field over M . Let ω_A and ω_{AB} be basic and connection forms with respect to the above orthonormal frames. Then, as is well known, we have

$$\begin{aligned} \omega_\alpha &= 0, & \omega_{i\alpha} &= \sum_j h_{ij}^\alpha \omega_j, & h_{ij}^\alpha &= h_{ji}^\alpha, \\ d\omega_i &= \omega_{ij} \wedge \omega_j, & i &\neq j, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, & \Omega_{ij} &= \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \\ R_{ijkl} &= \tilde{c}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \end{aligned}$$

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$$d\omega_{\alpha\beta} = \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum_{i,j} R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

$$R_{\alpha\beta ij} = \sum_k (h_{ik}^{\alpha} h_{kj}^{\beta} - h_{jk}^{\alpha} h_{ki}^{\beta}),$$

where we use the following convention on the ranges of indices:

$$1 \leq A, B, \dots \leq 2 + \nu, \quad 1 \leq i, j, \dots \leq 2, \quad 3 \leq \alpha, \beta, \dots \leq 2 + \nu.$$

M is said to be *minimal* if its mean curvature $\frac{1}{2} \sum_{i,\alpha} h_{ii}^{\alpha} e_{\alpha}$ vanishes identically, i.e., if $\text{trace } H_{\alpha} = 0$ for all α , $H_{\alpha} = (h_{ij}^{\alpha})$.

Let M_p be the *tangent space* at p to M and $N(p)$ the normal space of M at p . We can write *the (first) shape operator* (the second fundamental form) φ_1 as

$$\varphi_1(X, Y) = \sum h_{ij}^{\alpha} \omega_i(X) \omega_j(Y) e_{\alpha}, \quad X, Y \in M_p.$$

Now we define $h_{i_1 \dots i_k}$, $2 \leq k$, by

$$\sum h_{i_1 \dots i_k m}^{\alpha} \omega_m := dh_{i_1 \dots i_k}^{\alpha} + \sum h_{i_1 \dots i_{j-1} m i_{j+1} \dots i_k}^{\alpha} \omega_{m i_j} + \sum h_{i_1 \dots i_k}^{\beta} \omega_{\beta \alpha}.$$

Then we can define *the k th shape operator* (the k th second fundamental form) φ_k as the multilinear mapping from $M_p \times \dots \times M_p$ into $N(p)$ by

$$\varphi_k(X_1, X_2, \dots, X_{k+1}) := \sum h_{i_1 \dots i_{k+1}}^{\alpha} \omega_{i_1}(X_1) \cdots \omega_{i_{k+1}}(X_{k+1}) e_{\alpha},$$

where $X_j \in M_p$, $j = 1, 2, \dots, k+1$. We define the *k th normal space* $N_k(p)$ of M at p as

$$N_k(p) := \text{Span}\{\varphi_1(X_1, X_2), \varphi_2(X_1, X_2, X_3), \dots, \varphi_k(X_1, \dots, X_{k+1})\},$$

where $X_j \in M_p$, $j = 1, 2, \dots, k+1$. The immersion is said to be *m -regular* if $\dim N_k(p) > \dim N_{k-1}(p)$ at each point $p \in M$ for all $k = 1, 2, \dots, m$, where $\dim N_0(p) = 0$. A 0-regular mapping means an ordinary one, so an immersion is 0-regular.

2. Compact minimal positive curvature surfaces in a space form. Let M be a *compact connected oriented surface* which is *minimally* immersed in a $(2+\nu)$ -dimensional Riemannian manifold $\tilde{M}$ by the *m -regular* immersion. We suppose that *the Gaussian curvature K of M is positive*. Let U be a neighborhood of a point $p \in M$ in which there exist isothermal coordinates (u, v) and a frame field (e_1, e_2) such that

$$(2.1) \quad ds^2 = E\{du^2 + dv^2\}, \quad \omega_1 = \sqrt{E}du, \quad \omega_2 = \sqrt{E}dv,$$

where ds is the line element of M and $E = E(u, v)$ is a positive function on U . Since M is minimal in U , we may write

$$\omega_{1\alpha} = f_{\alpha}\omega_1 + g_{\alpha}\omega_2, \quad \omega_{2\alpha} = g_{\alpha}\omega_1 - f_{\alpha}\omega_2, \quad 2 < \alpha,$$

where f_{α} and g_{α} are functions on U . Then, using the structure equations, we easily see that the complex valued function

$$(2.2) \quad w_1(z, \bar{z}) = E^2(|G_1|^2 - |F_1|^2) + 2iE^2\langle G_1, F_1 \rangle$$

is holomorphic in $z = u + iv$, where $F_1 = \sum f_\alpha e_\alpha$ and $G_1 = \sum g_\alpha e_\alpha$. For a tangent vector $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$ of M_p , we have the mapping $\tilde{\varphi}_1$ from M_p into $N(p)$ as follows:

$$(2.3) \quad \tilde{\varphi}_1(X) := \varphi_1(X, X) = \cos 2\theta \cdot F_1 + \sin 2\theta \cdot G_1,$$

where $F_1 = \sum_\alpha f_\alpha e_\alpha = \sum_\alpha h_{11}^\alpha e_\alpha$ and $G_1 = \sum g_\alpha e_\alpha = \sum h_{12}^\alpha e_\alpha$. Then we have

LEMMA 1. *At each point p of M , the image of the tangent unit circle S_p^1 under the mapping $\tilde{\varphi}_1$ is a point or a circle according as $\dim N_1(p) = 0$ or $\neq 0$, where $S_p^1 = \{X \in M_p \mid |X| = 1\}$.*

PROOF. We easily see that $|w_1(z, \bar{z})|^2/E^4$ is a differentiable function on M . Since M is compact, $|w_1(z, \bar{z})|^2/E^4$ takes the maximum A at some point p_0 of M . If $A > 0$, then there exists a neighborhood U of p_0 in which $|w_1(z, \bar{z})|^2/E^4 > 0$, and there exist isothermal coordinates (u, v) and a frame field satisfying (2.1). Then from (2.2) we have

$$(2.4) \quad \Delta \log(|w_1(z, \bar{z})|^2/E^4) = -4\Delta \log E = 8EK, \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2,$$

because K is given by $K = -(1/2E)\Delta \log E$. It follows from $K > 0$ and (2.4) that the function $\tilde{w}_1 = \log(|w_1(z, \bar{z})|^2/E^4)$ is a subharmonic function on M . Since $\tilde{w}_1$ takes the maximum $\log A$ at $p_0 \in U$, it must be constant $\log A$ on U . Furthermore, since M is connected, $\tilde{w}_1$ is constant on M . It follows from this fact and (2.4) that K is identically zero on M , which contradicts $K > 0$. Thus the function $w_1(z, \bar{z}) = 0$, so F_1 and G_1 are orthonormal vectors in $N_1(p)$. Hence, we have proved our assertion.

If $m = 1$, that is, $\dim N_1(p) \neq 0$ at each point $p \in M$, then, by Lemma 1, we choose a neighborhood U of p in which there exist isothermal coordinates (u, v) and frame fields satisfying (2.1) and

$$(2.5) \quad \begin{aligned} \omega_{13} &= k_1 \omega_1 = \omega_{24}, & \omega_{1\beta} &= \omega_{2\beta} = 0, \\ \omega_{23} &= -k_1 \omega_2 = -\omega_{14}, & 4 &< \beta, \end{aligned}$$

where k_1 is a positive differentiable function on M . Using the structure equations, from (2.5) we have

$$(2.6) \quad \omega_{34} = 2\omega_{12} - (\log k_1)_2 \omega_1 + (\log k_1)_1 \omega_2,$$

where $d(\log k_1) = \sum_j (\log k_1)_j \omega_j$. Furthermore, by (2.5) we may put

$$(2.7) \quad k_1 \omega_{3\beta} = f_\beta \omega_1 + g_\beta \omega_2, \quad k_1 \omega_{4\beta} = g_\beta \omega_1 - f_\beta \omega_2, \quad 4 < \beta,$$

and define two normal vectors $F_2 = \sum f_\beta e_\beta$ and $G_2 = \sum g_\beta e_\beta$ on U . Then, from φ_2 we have the mapping $\tilde{\varphi}_2$ from M_p into $N_2(p)$ as

$$(2.8) \quad \tilde{\varphi}_2(X) := \varphi_3(X, X, X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,$$

for $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2 \in M_p$, $p \in U$. Using the structure equations, by (2.5) and (2.7) the complex valued function

$$w_2(z, \bar{z}) = E^3(|G_2|^2 - |F_2|^2) + 2iE^3 \langle F_2, G_2 \rangle$$

is holomorphic in z . Furthermore, $|w_2(z, \bar{z})|^2/E^6$ is a differentiable function on M . Hence, in the same way as Lemma 1, we can prove the following

LEMMA 2. *If $m = 1$, then, at each point $p \in M$, the image of a unit tangent circle S_p^1 under $\tilde{\varphi}_2$ is a point or a circle according as $\dim N_2(p) - \dim N_1(p) = 0$ or $\neq 0$.*

If $m = 2$, that is, $\dim N_2(p) - \dim N_1(p) > 0$ at each point $p \in M$, then, by Lemma 1 and Lemma 2, we choose a neighborhood U of p in which there exist isothermal coordinates (u, v) and frame fields satisfying (2.1), (2.5) and

$$(2.9) \quad \begin{aligned} k_1\omega_{35} &= k_2\omega_1 = k_1\omega_{46}, & \omega_{3\gamma} &= \omega_{4\gamma} = 0, \\ k_1\omega_{36} &= k_2\omega_2 = -k_1\omega_{45}, & 6 &< \gamma, \end{aligned}$$

where k_2 is a positive differentiable function on M . Let $d(\log k_2) = \sum (\log k_2)_j \omega_j$; then from (2.9) we have

$$(2.10) \quad \omega_{56} = 3\omega_{12} - (\log k_2)_2 \omega_1 + (\log k_2)_1 \omega_2$$

and we may write

$$(2.11) \quad k_2\omega_{5\gamma} = f_\gamma\omega_1 + g_\gamma\omega_2, \quad k_2\omega_{6\gamma} = g_\gamma\omega_1 - f_\gamma\omega_2, \quad 6 < \gamma.$$

From the third shape operator φ_3 we have the mapping $\tilde{\varphi}_3$ from M_p into $N_3(p)$ as

$$\tilde{\varphi}_3(X) := \varphi_3(X, X, X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,$$

for $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2 \in M_p$, $p \in M$, where $F_3 = \sum f_\gamma e_\gamma$ and $G_3 = \sum g_\gamma e_\gamma$ are normal vector fields on U . Furthermore, from (2.9), (2.10) and (2.11) we see that the complex function

$$\omega_3(z, \bar{z}) = E^4(|G_3|^2 - |F_3|^2) + 2iE^4\langle F_3, G_3 \rangle$$

is holomorphic in z .

Continuing this way, we have the following:

THEOREM 1. *Let M be a connected compact oriented surface of positive curvature minimally immersed in a $(2 + \nu)$ -dimensional space form $\bar{M}$ of constant curvature c . If the immersion is m -regular, then, at each point on M , the image of a unit tangent circle S_p^1 under the mapping $\tilde{\varphi}_n$ (the n th shape operator) is a circle for any $n = 1, 2, \dots, m$, and the image of S_p^1 under $\tilde{\varphi}_{m+1}$ is a point or a circle according as $\dim N_{m+1}(p) - \dim N_m(p) = 0$ or $\neq 0$.*

PROOF. For $n = 1, 2$, we have proved our assertion in Lemma 1 and Lemma 2. By induction on n , we will prove our assertion. So we assume that the above assertion holds for all $t \leq n - 1 < m, 3 \leq n$. Then we can choose a neighborhood U of a point $p \in M$ in which there exist isothermal coordinates (u, v) and frame fields satisfying (2.1) and

$$(2.12)_{n-1} \quad \begin{cases} k_{t-1}\omega_{\alpha_1\beta_1} = k_t\omega_1 = k_{t-1}\omega_{\alpha_2\beta_2}, & \omega_{\alpha_1\gamma} = \omega_{\alpha_2\gamma} = 0, \\ k_{t-1}\omega_{\alpha_1\beta_2} = k_t\omega_2 = -k_{t-1}\omega_{\alpha_2\beta_1}, & 2t + 2 < \gamma, \\ \alpha_1 = 2t - 1, & \alpha_2 = 2t, & \beta_1 = 2t + 1, & \beta_2 = 2t + 2, \\ & & t = 1, 2, \dots, n - 1, \end{cases}$$

where $k_0 = 1$ and k_t ($1 \leq t \leq n - 1$) are positive differentiable functions on M . Using the structure equations, from (2.12) $_{n-1}$ we have

$$(2.13)_{n-1} \quad \omega_{\beta_1\beta_2} = (t + 1)\omega_{12} - (\log k_t)_2 \omega_1 + (\log k_t)_1 \omega_2,$$

where $\beta_1 = 2t + 1$, $\beta_2 = 2t + 2$, and $d(\log k_t) = \sum (\log k_t)_j \omega_j$ for $t = 1, 2, \dots, n - 1$. Furthermore, from $(2.12)_{n-1}$ we may write

$$\begin{aligned} k_{n-1} \omega_{a_1 \gamma} &= f_\gamma \omega_1 + g_\gamma \omega_2, & a_1 &= 2n - 1, \\ k_{n-1} \omega_{a_2 \gamma} &= g_\gamma \omega_1 - f_\gamma \omega_2, & a_2 &= 2n, & 2n < \gamma. \end{aligned}$$

From the n th shape operator φ_n we have the mapping $\tilde{\varphi}_n$ from M_p into $N_n(p)$ as $\tilde{\varphi}_n(X) := \tilde{\varphi}_n(X, \dots, X)$ for $X \in M_p$. For a unit tangent vector $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$ at p to M , we have

$$\tilde{\varphi}_n(X) = \cos(n+1)\theta \cdot F_n + \sin(n+1)\theta \cdot G_n,$$

where $F_n = \sum f_\gamma e_\gamma$ and $G_n = \sum g_\gamma e_\gamma$ are normal vector fields on U . Using the structure equations, from $(2.12)_{n-1}$ and $(2.13)_{n-1}$ we see that the complex valued function

$$(2.14) \quad w_n(z, \bar{z}) = E^{n+1}(|G_n|^2 - |F_n|^2) + 2iE^{n+1}\langle G_n, F_n \rangle$$

is holomorphic in $z = u + iv$. As stated above, we easily see that

$$|w_n(z, \bar{z})|^2 / E^{2n+2} = \{|G_n|^2 - |F_n|^2\}^2 + 4\langle G_n, F_n \rangle^2$$

is a differentiable function on M . Hence, in the same way as in the proof of Lemma 1, at each point $p \in M$, the image of S_p^1 under $\tilde{\varphi}_n$ is a circle, because $n \leq m$. Furthermore, when $n = m + 1$, by the above consideration we see that the image of S_p^1 under $\tilde{\varphi}_{m+1}$ is a point or a circle according as $\dim N_{m+1}(p) - \dim N_m(p) = 0$ or $\neq 0$. Thus we have proved our assertion.

Now we can prove the following:

THEOREM 2. *Let M be a connected compact oriented surface minimally immersed in a $(2 + \nu)$ -dimensional space form $\tilde{M}$ of constant curvature c . If the immersion is m -regular and the Gaussian curvature K satisfies the inequality*

$$\frac{2c}{(m+2)(m+3)} \leq K \leq \frac{2c}{(m+1)(m+2)},$$

then M is of constant curvature $2c/((m+2)(m+3))$ or $2c/((m+1)(m+2))$.

PROOF. First we consider the case that $\dim N_{m+1}(p) = \dim N_m(p)$ at each point $p \in M$, so $k_{m+1}^2 = 0$ on M by Theorem 1. Using the structure equations, from (2.5), (2.12) and (2.13) we have

$$(2.15) \quad \begin{aligned} \Delta \log k_t &= E\{(t+1)K - 2k_t^2/k_{t-1}^2 + 2k_{t+1}^2/k_t^2\}, & t &= 1, 2, \dots, m-1, \\ \Delta \log k_m &= E\{(m+1)K - 2k_m^2/k_{m-1}^2\}, & \Delta &= \partial^2/\partial u^2 + \partial^2/\partial v^2, \end{aligned}$$

which imply

$$(2.16) \quad \Delta \log(k_1 \cdot k_2 \cdots k_m) = E\left\{\frac{m(m+3)}{2}K - 2k_1^2\right\} = E\left\{\frac{(m+1)(m+2)}{2}K - c\right\},$$

because $K = c - 2k_1^2$. Since $K \leq 2c/((m+1)(m+2))$, (2.16) implies that the function $\log(k_1 \cdots k_m)$ is a superharmonic function on M . Since M is compact, $\log(k_1 \cdots k_m)$ must be constant on M , which, together with (2.16), implies $K = 2c/((m+1)(m+2))$ identically on M .

Next, we consider the case that there exists a point $p_0 \in M$ such that $\dim N_{m+1}(p_0) > \dim N_m(p_0)$. We choose a neighborhood U of p_0 in which there exist isothermal coordinates (u, v) and frame fields satisfying (2.1) and (2.12) $_{m+1}$. We must remark that k_{m+1}^2 is a differentiable function on M and vanishes at a point where $\dim N_{m+1}(p) = \dim N_m(p)$. On U we can consider the image of S_p^1 under the mapping $\tilde{\varphi}_{m+2}$. For the same reason as above, it is a point or a circle according as $\dim N_{m+2}(p) - \dim N_{m+1}(p) = 0$ or $\neq 0$. Using the structure equations, from (2.5), (2.12) $_{m+1}$ and (2.13) $_{m+1}$ we have

$$(2.17) \quad \Delta \log k_t = E\{(t+1)K - 2k_t^2/k_{t-1}^2 + 2k_{t+1}^2/k_t^2\}, \quad t = 1, \dots, m+1,$$

where $k_{m+2}^2 = \sum_{2(m+2) < \gamma} f_\gamma^2$ is the square of the radius of the image of S_p^1 . From (2.16) we have

$$(2.18) \quad \begin{aligned} \Delta(\log k_1 \cdots k_{m+1}) &= E \left\{ \frac{(m+1)(m+4)}{2} K - 2k_1^2 + 2k_{m+2}^2/k_{m+1}^2 \right\} \\ &\geq E \left\{ \frac{(m+2)(m+3)}{2} K - c \right\}, \end{aligned}$$

which, together with $K \geq 2c/((m+2)(m+3))$, implies $\log(k_1 \cdots k_{m+1})$ is a subharmonic function on U . Here we may assume that the differentiable function $k_1^2 \cdots k_{m+1}^2$ takes the maximum value at p_0 . Then $\log(k_1 \cdots k_{m+1})$ takes its maximum at p_0 in U , so it must be constant on U . Hence (2.18) implies that $K = 2c/((m+2)(m+3))$ on U . Since M is connected, $K = 2c/((m+2)(m+3))$ on M . Thus we have proved our assertion.

We see that our main theorem is obtained as a corollary of the results in [3] and the above Theorem 2.

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