A PROOF OF THE ISOSYSTOLIC INEQUALITY FOR THE KLEIN BOTTLE

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Dedicated to Professor S. Murakami on his sixtieth birthday

ABSTRACT. We give a simple proof of the isosystolic inequality for the Klein bottle, which was first obtained by C. Bavard.

For a smooth riemannian metric g on the Klein bottle K we consider the metric invariant $\text{Sys Ar}(g) := \text{Area}(K, g) / \text{sys}^2(K, g)$, where $\text{sys}(K, g)$ denotes the infimum of the g -length of homotopically nontrivial closed curves on K . Then C. Bavard obtained the following remarkable inequality.

THEOREM [1]. $\text{Sys Ar}(g) \geq 2\sqrt{2}/\pi$, where the equality holds for the metric of positive constant curvature with singularities.

Bavard used the result of Pu [3] on the geometry of the Möbius band. On the other hand there is also a paper by C. Blatter on the geometry of the Möbius band, and in this note I would like to point out that the above theorem follows directly from the Blatter results.

First, we recall the results of Blatter: Let $\mathbf{S}^2 \subset \mathbf{R}^3$ be the unit sphere and (ϑ, φ) be the Fermi coordinates with respect to the equator $\mathbf{S}^1$. Namely for $m \in \mathbf{S}^2 \setminus \{\text{north and south poles}\}$ we put $\vartheta(m) := \text{dist}_{\mathbf{S}^2}(m, \mathbf{S}^1)$ and $\varphi(m)$ denotes the angular coordinate of the footpoint $p(m)$ of m on $\mathbf{S}^1$ with $\text{dist}_{\mathbf{S}^2}(m, p(m)) = \text{dist}_{\mathbf{S}^2}(m, \mathbf{S}^1)$. Now we consider the zone $Z_{\vartheta_0} := \{m \in \mathbf{S}^2; |\vartheta(m)| < \vartheta_0\}$ ($0 < \vartheta_0 < \pi/2$) and the isometric involution T on Z_{ϑ_0} defined by $T(m) := -m$. The Uniformization Theorem implies that for any Riemannian structure (M, g) on the Möbius band there exist $0 < \vartheta_0 < \pi/2$ and a positive function $\bar{\rho}$ on Z_{ϑ_0} invariant by T , so that (M, g) is isometric to $(M_{\vartheta_0}, \rho^2 g_0) := (Z_{\vartheta_0}, \bar{\rho}^2 g_0) / \{T\}$, where g_0 denotes the metric of constant curvature 1. Now let Γ be a generator of $\pi_1(M) \cong \mathbf{Z}$, which is represented by great circle segments joining m and $T(m)$ and we put $l(g) := \inf\{\text{length}_g \gamma; \gamma \in \Gamma\}$. Next let Γ^* be the family of curves in M defined by $\varphi = \text{constant}$ connecting boundary points, which are transversal to the curves γ in Γ and have the intersection number ± 1 with γ . We also put $l^*(g) := \inf\{\text{length}_g \gamma^*; \gamma^* \in \Gamma^*\}$. Then C. Blatter obtained the following:

PROPOSITION [2]. (1) $\text{Area}(M, g) / l^2(g) \geq 2/\pi \cdot \sin \vartheta_0$, where the equality holds iff ρ is equal to a constant.

(2) $\text{Area}(M, g) / l(g)l^*(g) \geq \sin \vartheta_0 / \vartheta_0$ if $0 < \vartheta_0 < \Theta$, where $0 < \Theta < \pi/2$ is given by $\tan \Theta = 2\Theta$. The equality holds iff ρ is equal to a constant.

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Note that $\Theta > \pi/3$ for above Θ . Now we shall give our proof of the theorem:

1. Given (K, g) , there exists $0 < \vartheta_0 < \pi/2$ such that (K, g) is the double of (M_{ϑ_0}, g_1) and (M_{ϑ_0}, g_2) with $g_i = \rho_i^2 g_0$ ($i = 1, 2$) and $\rho_1|_{\partial M_{\vartheta_0}} = \rho_2|_{\partial M_{\vartheta_0}}$. In fact, (K, g) is conformal to a flat $(K, \bar{g}_0)$ which is obtained from a rectangle in $\mathbf{R}^2$. Then $(K, \bar{g}_0)$ is the double of the flat Möbius band $(M, \tilde{g}_0)$, which is conformal to (M_{ϑ_0}, g_0) for some ϑ_0 . Since $\varphi \rightarrow \varphi + \alpha$ ($\alpha \in \mathbf{S}^1$) are isometries with respect to the metric $\bar{g}_0$ on K , defined by taking $\rho_i = 1$ ($i = 1, 2$), we may assume that ρ_i depend only on ϑ by the standard argument averaging ρ_i by the action of $\mathbf{S}^1$ (see [2, 3]). Then we have easily

$$\text{Area}(K, g) = \text{Area}(M_{\vartheta_0}, g_1) + \text{Area}(M_{\vartheta_0}, g_2)$$

and

$$\text{sys}(K, g) \leq \min\{l(g_1), l(g_2), l^*(g_1) + l^*(g_2)\}.$$

2. (Case when $\pi/2 > \vartheta_0 \geq \pi/4$). We have from (1) of the Proposition

$$\begin{aligned} \text{Sys Ar}(K, g) &\geq \text{Area}(M_{\vartheta_0}, g_1)/l^2(g_1) + \text{Area}(M_{\vartheta_0}, g_2)/l^2(g_2) \\ &\geq 2 \cdot 2/\pi \cdot \sin \vartheta_0 \geq 4/\pi \cdot \sin \pi/4 = 2\sqrt{2}/\pi, \end{aligned}$$

where the equality holds iff $\vartheta_0 = \pi/4$ and $\rho_1 = \rho_2 = \text{constant}$.

3. (Case when $0 < \vartheta_0 \leq \pi/4$). We have from (2) of the Proposition,

$$\begin{aligned} \text{Sys Ar}(K, g) &\geq \{\text{Area}(M_{\vartheta_0}, g_1) + \text{Area}(M_{\vartheta_0}, g_2)\} / \{\text{sys}(K, g)(l^*(g_1) + l^*(g_2))\} \\ &\geq \min\{\text{Area}(M_{\vartheta_0}, g_1)/l(g_1)l^*(g_1), \text{Area}(M_{\vartheta_0}, g_2)/l(g_2)l^*(g_2)\} \\ &\geq \sin \vartheta_0 / \vartheta_0 \geq (\sin \pi/4)/(\pi/4) = 2\sqrt{2}/\pi, \end{aligned}$$

because $\vartheta \rightarrow \sin \vartheta / \vartheta$ is strictly monotone decreasing on $(0, \pi/2]$. The equality holds iff $\vartheta_0 = \pi/4$ and $\rho_1 = \rho_2 = \text{constant}$.

4. It is easy to see that for the Riemannian metric $(K, \bar{g}_0)$ obtained in the above by the condition $\vartheta_0 = \pi/4$ and $\rho_1 = \rho_2 = \text{constant}$, $\text{Sys Ar}(K, \bar{g}_0)$ is in fact equal to $2\sqrt{2}/\pi$. Note that $\bar{g}_0$ is a continuous Riemannian metric and smooth except at ∂M_{ϑ_0} .

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