THE FIRST DIAMETER OF 3-MANIFOLDS OF POSITIVE SCALAR CURVATURE

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(Communicated by David G. Ebin)

ABSTRACT. We seek a universal upper bound for the first diameter of 3-manifolds of scalar curvature $\geq +1$. We find it in the case of finite fundamental group by using a combinatorial theorem about finite trees, and in the case when π_1 is infinite cyclic by using a weak notion of a π_1 -equivariant Busemann function.

1. The classical theorem of Myers asserts that if the Ricci curvature of all unit vectors on an n -dimensional compact Riemannian manifold X is $\geq n - 1$, then its diameter is $\leq \pi$ [2, p. 27]. On the other hand, for $n \geq 3$ Gromov and Lawson [3] construct a metric of positive *scalar* curvature on the connected sum of manifolds of positive scalar curvature. Thus if Γ is any graph with even valence at each vertex, one can take $S^2 \times \Gamma$ and smooth $S^2 \times \{\text{neighborhoods of vertices}\}$ to obtain a manifold of positive scalar curvature with arbitrarily large diameter. Nonetheless they show in [4] that in a sense this is the 'worst' that can happen in the 3-dimensional case, at least if the manifold is simply connected.

Before stating their result, we will define a convenient metric invariant.

1.1. The *first diameter* of X , denoted $\text{diam}_1 X$, is the infimum of numbers $\varepsilon > 0$ such that there exists a map from X to a metric graph with the following property: the inverse image of any point on the graph has diameter $\leq \varepsilon$ [5, p. 126]. A manifold with small first diameter looks like a graph to a nearsighted observer.

In [4, p. 386] the authors show the following.

1.2. In a 3-dimensional manifold X with scalar curvature $\geq +1$, every contractible loop must already bound in its 2π -neighborhood.

The idea of the proof is to span the curve by a stable minimal surface and apply the stability inequality in the form first used in [6, p. 139] to a cleverly chosen variation of the surface.

The authors then consider the metric graph whose points are connected components of the levels of the distance function from a point, and derive from 1.2 the following estimate [4, Corollary 10.11] (see also [1]).

1.3. If X is simply connected then $\text{diam}_1 X \leq 12\pi$.

The corresponding graph in such case is a tree. It would be desirable to remove the simple connectivity assumption in 1.3.

Note that the topology in this situation is rather well understood. Namely, a 3-manifold with a metric of positive scalar curvature cannot contain $K(\pi, 1)$ factors in

Received by the editors September 10, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C21; Secondary 57M99, 05C05.

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 0002-9939/88 \$1.00 + \$.25 per page

its prime factor decomposition. Given the general feeling that the scalar curvature of X is very flabby, what can be said about its global geometry?

It is not obvious that if $\tilde{X}$ is the universal cover of X , then $\text{diam}_1 X \leq \text{diam}_1 \tilde{X}$. One might expect that if $\tilde{X}$ looks like a tree, then X should look like a quotient of that tree. However, the quotient is not obviously defined since the tree is not π_1 -equivariant: its construction involved choosing a point in $\tilde{X}$. It is also clear that if an equivariant tree exists, it could not be a subset of $\tilde{X}$ due to possible torsion in $\pi_1(X)$. The next logical step seems to be to use the standard imbedding of $\tilde{X}$ in the space $L^\infty(\tilde{X})$ of functions on $\tilde{X}$, and look for the tree there. Unfortunately, this approach has not yet succeeded.

We are able to show that $\text{diam}_1 X \leq 200\pi$ in the case when $\pi_1(X)$ is finite by observing that the tree-like universal cover $\tilde{X}$ must possess a smeared center (Theorem 3.1).

We show in §4 that a sufficiently ‘long’ (see 4.2) element $g \in \pi_1(X)$ of infinite order has a weak Busemann function on $\tilde{X}$ which is g -equivariant, whose gradient may be large but whose level sets have controlled diameter. This allows us to prove that if $\pi_1(X) = Z$ then $\text{diam}_1 X \leq 100\pi$ (Theorem 4.1).

In §2 we prove two lemmas which are useful for both of our first diameter estimates. In §3 we prove the finite fundamental group estimate, and in §4 the infinite cyclic fundamental group estimate.

ACKNOWLEDGEMENT. The author is grateful to H. B. Lawson for a number of stimulating discussions.

2.

2.1 LEMMA. *Let X be a simply connected 3-manifold with scalar curvature $\geq +1$. Let $Y \subset X$ be a connected subset. Consider the distance function $\text{dist}_Y(x) = \text{dist}(x, Y)$. Then a connected component of a level set $\text{dist}_Y = d$, where $d > 4\pi$, has diameter $\leq 12\pi$.*

PROOF. Let x_1 and x_2 belong to the same connected component A of the level $\text{dist}_Y(x) = d$. Choose points $y_i \in Y$ with $\text{dist}(x_i, y_i) = d$, $i = 1, 2$. Choose a path $c \subset Y$ joining y_1 and y_2 , and a path $b \subset A$ joining x_2 and x_1 . Denote by $y_i x_i$ a shortest path joining y_i with x_i . The closed loop $\gamma = c \cup y_2 x_2 \cup b \cup (y_1 x_1)^{-1}$ bounds a surface $D \subset X$ in the 2π -neighborhood of γ by 1.2. Triangulate D into simplices of diameter $\leq \varepsilon$. Send each vertex of the triangulation to a nearest point of γ . If $d > 4\pi + \varepsilon$, then a vertex sent to c and a vertex sent to b are never adjacent. We send each edge of the 2-skeleton of D to a path in γ (thought of as a quadrilateral with vertices x_1, x_2, y_2, y_1) in accordance with the following 3-step procedure:

1. If the endpoints are sent to the same side of γ , join them by a path contained in that side.
2. If the endpoints are sent to adjacent sides of γ , join them by a path via the common vertex.
3. If the endpoints are sent to opposite sides $x_1 y_1$ and $x_2 y_2$, join them via the side c .

It is clear that the only time the boundary of a triangle is sent to γ with nonzero degree is when the vertices p, q, r are sent, respectively, to the sides $y_1 x_1, b, y_2 x_2$ (up to reordering). Since D cannot be retracted to γ , such a triangle must exist.

To summarize, there exist points $p, q, r \in D$ with pairwise distances $\leq \varepsilon$, and points $p_0 \in y_1x_1$, $q_0 \in b$, $r_0 \in y_2x_2$ such that the distances $p_0p, q_0q, r_0r \leq 2\pi$. Meanwhile, $y_1p_0 \geq y_1q_0 - p_0q_0 \geq y_1q_0 - (4\pi + \varepsilon) \geq d - (4\pi + \varepsilon)$, hence $x_1p_0 = y_1x_1 - y_1p_0 \leq d - (d - (4\pi + \varepsilon)) = 4\pi + \varepsilon$, and $x_1x_2 \leq x_1p_0 + p_0r_0 + r_0x_2 \leq 12\pi + 3\varepsilon$. Lemma 2.1 is proved.

2.2. Let us introduce some convenient terminology and notation. Suppose X is simply connected and let $Y \subset X$ be a connected subset. Define a stratum to be a connected component of a level set of the distance function from Y . Define an equivalence relation among points of X by setting $x \sim x'$ if x and x' lie in the same stratum. Let $T(Y) = X/\sim$. Then $T(Y)$ is a tree by the covering homotopy property. Let dist_T denote the metric in $T(Y)$. Let $f: X \rightarrow T(Y)$ be the projection. It is clear that if Y is bounded, f gives a one-to-one correspondence between the ends at infinity of X and those of $T(Y)$. We will typically use a tree $T(p)$, $p \in X$. Note that $\text{dist}_T(f(x), f(y)) \leq \text{dist}(x, y) \leq \text{dist}_T(f(x), f(y)) + 12\pi$ if Y is a point, by Lemma 2.1.

The following estimate will be used in the proof of both Theorems 3.1 and 4.1.

2.3. LEMMA. *Let X be a simply connected 3-manifold of scalar curvature $\geq +1$. Let $L \subset X$ be any minimizing arc between $x, y \in X$. Let $\bar{L} \subset T(p)$ be the imbedded interval with endpoints $f(x)$ and $f(y)$. Let $U_r(\bar{L}) \subset T(p)$ be the r -neighborhood of $\bar{L}$. Then $f(L) \subset U_{6\pi}(\bar{L})$.*

PROOF. Clearly $\bar{L} \subset f(L)$. Suppose for $z \in L$ we have $\text{dist}_T(f(z), \bar{L}) > 6\pi$. Then there are points z_1 and z_2 on L , one to the right and the other to the left of z , such that $f(z_1) = f(z_2) =$ the point of $\bar{L}$ nearest $f(z)$. We have

$$\begin{aligned} \text{dist}(z_1, z_2) &= \text{dist}(z_1, z) + \text{dist}(z, z_2) \\ &\geq \text{dist}_T(f(z_1), f(z)) + \text{dist}_T(f(z), f(z_2)) \\ &> 12\pi, \end{aligned}$$

in contradiction to the proof of 1.3 given in [4].

3.

3.1 THEOREM. *Let X be a compact 3-manifold of scalar curvature $\geq +1$. Suppose $\pi_1(X)$ is finite. Then $\text{diam}_1 X \leq 200\pi$.*

PROOF. The midpoint of a longest imbedded interval in a tree will be called its *center*. The following fact is obvious.

3.2 LEMMA. *Let c be the center of a metric tree T with diameter d . Suppose $t \in T$ is a point lying in some imbedded interval $[a, b] \subset T$ so that $\text{dist}(t, a)$ and $\text{dist}(t, b)$ are both $\geq d/2 - \varepsilon$. Then $\text{dist}(t, c) \leq \varepsilon$.*

Take a pair of points $a, b \in \tilde{X}$ with $\text{dist}(a, b) = \text{diam } \tilde{X}$. Let $x \in \tilde{X}$ be the midpoint of a shortest arc joining a and b . Let $f: \tilde{X} \rightarrow T(p)$ be as in 2.2. Let $L \subset T(p)$ be the imbedded path joining $f(a)$ and $f(b)$. Let $t \in L$ be the point of L nearest to $f(x)$. By Lemma 2.3, $\text{dist}_T(f(x), t) \leq 6\pi$. Thus we have

$$\begin{aligned} \text{dist}_T(t, f(a)) &\geq \text{dist}_T(f(x), f(a)) - 6\pi \geq \text{dist}(x, a) - 24\pi - 6\pi \\ &= \frac{1}{2}\text{diam } X - 30\pi \geq \frac{1}{2}\text{diam } T - 30\pi. \end{aligned}$$

By Lemma 3.2, $\text{dist}_T(t, c) \leq 30\pi$ and so $\text{dist}_T(f(x), c) \leq 36\pi$. Clearly if $g \in \pi_1(X)$ then also $\text{dist}_T(f(gx), c) \leq 36\pi$. Hence $\text{dist}_T(f(gx), f(x)) \leq 72\pi$ and $\text{dist}(x, gx) \leq 72\pi + 24\pi$. Thus the orbit of $x \in \tilde{X}$ under $\pi_1(X)$ has diameter at most 96π .

Let $Y \subset X$ be the union of all minimizing geodesics joining pairs of points from this orbit. It is clear that $\text{diam } Y \leq 192\pi$. By Lemma 2.1, any stratum of dist_Y has diameter $\leq 192\pi + 8\pi$. Since Y is invariant under the action of the fundamental group, the stratification descends to X . The theorem is proved.

4.

4.1 THEOREM. *Let X be a complete 3-manifold of scalar curvature $\geq +1$. Suppose $\pi_1(X)$ is infinite cyclic. Then $\text{diam}_1 X \leq 100\pi$.*

PROOF. Let $L \subset \tilde{X}$ be the lift to the universal cover of a shortest loop in the free homotopy class of the generator $g \in \pi_1(X)$.

4.2. If the loop has length $\leq 92\pi$, we apply Lemma 2.1 with $Y = L$ and then descend the resulting stratification to X using the fact that Y is invariant under $\pi_1(X)$. Then all strata have diameter $\leq 100\pi$.

If the loop has length $> 92\pi$, then only the strata at least 4π away from L have small diameter. We first chop off these faraway branches of the tree-like manifold X and then stratify the neighborhood $U_{4\pi}(L)$ by inverse images of points of L under a suitable projection. This must be done in a way compatible with the stratification of the chopped-off branches.

Everything that follows is done in a π_1 -equivariant way. Replace the distance function to L by a smooth approximation such that 4π is a regular value. Triangulate the boundary of $U_{4\pi}(L)$ and extend to a triangulation of $U_{4\pi}(L)$.

Send each vertex of the triangulation to a nearest point of L , except that vertices of a connected component of the boundary should be sent to the same point. Since the components have diameter $\leq 12\pi$, we may assume that every vertex is sent to a point of L at distance $\leq 12\pi + 4\pi = 16\pi$. Extend linearly to each simplex to get a map $h: U_{4\pi}(L) \rightarrow L$. Suppose two neighboring vertices v and w with $\text{dist}(v, w) \leq \varepsilon$ are sent to points a and b . Then $\text{dist}(a, b) \leq 32\pi + \varepsilon$. We must verify that L realizes the distance between a and b . This is hardly surprising since intervals along L are minimizing up to at least 92π , but the proof is surprisingly hard.

Among all pairs of points on L with distance $\leq 32\pi + \varepsilon$ and such that L does not realize the distance between them, choose points $a', b' \in L$ so as to minimize $\text{dist}_L(a', b')$. Clearly $\text{dist}_L(a', b') > 92\pi$. Let $m \in L$ be the point halfway between a' and b' . Join a' and b' by a shortest arc. The resulting triangle $a'b'm$ in $\tilde{X}$ is a closed loop which must bound in its 2π -neighborhood by 1.2. Arguing as in 2.1, we get points $p_0 \in a'm$, $q_0 \in mb'$, $r_0 \in b'a'$ with pairwise distances $\leq 4\pi + \varepsilon$. We may assume without loss of generality that r_0 is closer to a' than to b' . Because of our choice of a' and b' , L must realize the distance between p_0 and q_0 . Hence

$$\begin{aligned} \text{dist}(m, a') &\leq \text{dist}(m, p_0) + \text{dist}(p_0, r_0) + \text{dist}(r_0, a') \\ &\leq 4\pi + \varepsilon + 4\pi + \varepsilon + \frac{1}{2}\text{dist}(a', b') \\ &\leq 8\pi + 2\varepsilon + 16\pi + \varepsilon < 32\pi + \varepsilon, \end{aligned}$$

while $\text{dist}_L(m, a') = \frac{1}{2}\text{dist}_L(a', b') \geq 46\pi$. This contradicts our choice of a', b' .

Thus the retraction of every simplex to L has length $\leq 32\pi + \varepsilon$. Let $y \in L$. We estimate the diameter of $h^{-1}(y)$. Let $x_1, x_2 \in h^{-1}(y)$. Let v_i be a vertex of the simplex containing x_i , $i = 1, 2$. Then

$$\begin{aligned} \text{dist}(v_1, v_2) &\leq \text{dist}(v_1, h(v_1)) + \text{dist}(h(v_1), y) + \text{dist}(y, h(v_2)) + \text{dist}(h(v_2), v_2) \\ &\leq 16\pi + (32\pi + \varepsilon) + (32\pi + \varepsilon) + 16\pi \\ &= 96\pi + 2\varepsilon, \end{aligned}$$

and $\text{dist}(x_1, x_2) \leq \text{dist}(v_1, v_2) + 2\varepsilon \leq 100\pi$.

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