INDEPENDENT STATEMENTS ABOUT FINITE STRETCHABLE STRUCTURES

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ABSTRACT. We consider several analogs of the theorem of Ehrenfeucht and Mostowski, in which the models generated by indiscernibles are finite; we show these analogs to be independent from strong axioms, ranging from Peano's arithmetic to set theory with Mahlo cardinals.

0. The basic idea. Models generated by enough indiscernibles can be "stretched", as first done by Ehrenfeucht and Mostowski, [E-M]. In particular, stretching *finite* models of this kind may turn out to be a useful way to apply infinite combinatorics to finite objects, so to speak: the idea is that you stretch your finite model M , to get an infinite structure M_∞ ; then you apply infinite combinatorics to M_∞ , obtaining an extension N_∞ , and finally you go back to finite objects, by taking a suitable restriction N of N_∞ . Hopefully, N is a useful extension of M , which it would have been difficult to obtain without the infinitary detour.

The result of this paper (Theorem 5) illustrates the above idea, for it shows that the existence of many a finite stretchable structure N requires strong axioms to be proved; and this means that some kind of infinitary detour like N_∞ is needed to construct N . As to the usefulness of these structures N , we have no proof for it but it seems reasonable to expect it.

The numbering convention adopted here is: numbers always increase, so that for instance we will write

"(4) *Remarks.* (a)
 (b)"

(5) *Theorem.*"

and later refer to this as Remark 4.b, Theorem 5, etc., regardless of the number of preceding remarks and theorems.

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1. Definitions. Indiscernibles and stretchings. (a) We say that X is (*order-*)*indiscernible* in the structure M iff X is a subset of $|M|$ with a linear ordering denoted $<$, and all $<$ -isomorphic sequences from X satisfy in M the same *atomic* sentences.

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(b) If in addition X generates the structure M , then we can *stretch* M , at least if X is infinite; that is: define for every linear ordering Y a structure $M(Y)$, generated by Y and such that any two $<$ -isomorphic sequences from X and Y respectively satisfy the same atomic sentences, in M and $M(Y)$ respectively. $M(Y)$ is unique up to isomorphism.

(c) If M , hence X are finite, then we still can define these stretchings $M(Y)$, provided $\text{card } X \geq n \cdot \max(p, q + 1)$, where p (resp. q) is the largest number of arguments of the relation (resp. function) symbols of the language of M ; and n is the smallest integer such that all elements of M are generated by at most n elements of X .

We call *stretchable structure* any (finite or infinite) structure M satisfying either (b) if M is infinite or (c) if M is finite.

The basic result about these structures is of course the theorem of Ehrenfeucht and Mostowski; we next state a part of this theorem in a way that prepares the sequel:

For any universal sentence ψ ,

(A) ψ has an infinite model $\Leftrightarrow$ it has a stretchable model.

In most cases, the stretchable model of φ in (A) will be infinite, so that the $\Leftarrow$ part of (A) has little content; but if we restrict considerably the expressive power which φ is allowed to have, then we obtain variants of (A) in which only finite stretchable structures are used (so that their stretching is needed to prove the $\Leftarrow$ part of the variant). Here is the simplest:

We let P be a predicate symbol; $\forall x_1 \cdots x_n \in P$ will abbreviate the prefix $\forall x_1 : P(x_1) \cdots \forall x_n : P(x_n)$. L is any language which does *not* contain P . Then for every open formula $\varphi(x_1 \cdots x_n)$ of L ,

(B) $\forall x_1 \cdots x_n \in P \varphi$ has a model in which P is infinite $\Leftrightarrow$ it has a finite stretchable model in which P is a set of $\geq n$ indiscernibles.

2. Remark. The case of the formula $\varphi(x_1 \cdots x_n) = \bigvee \{x_i = x_j; i \neq j\}$ shows the necessity of requiring at least n elements in P . The proof of (B) and of other variants (C)–(E) is given in (7) below.

The sentences $\forall \bar{x} \in P \varphi$ are so poor that (B) is no surprise; more surprising will be that, poor as they are, these sentences allow variants of (B) which are independent of strong axioms. These variants arise by considering stretchings which generate well ordered models.

3. Definition. Special and remarkable indiscernibles. We assume that $<$ is in the language of our structures, and is interpreted by a linear ordering; for every ordinal α , we call α -models those in which $<$ is of type α . We say that a set X of indiscernibles in the structure M is *special* (resp. *remarkable*; we also say that the elements of X are special or remarkable) iff for all terms $s(v_1 \cdots v_n)$, $t(v_1 \cdots v_{n+p})$ and all $x_1 < \cdots < x_{n+p}$ in X ,

(i) $M \models s(x_1 \cdots x_n) < x_{n+1}$,

(ii) $M \models t(x_1 \cdots x_{n+p}) < x_n \rightarrow t(x_1 \cdots x_{n+p}) = t(x_1 \cdots x_n y_1 \cdots y_p)$ [resp. $M \models \cdots < x_{n+1} \cdots$] whenever $x_{n+1} \leq y_1 < \cdots < y_p$ in X .

4. Fact. Let M be a finite stretchable structure; the indiscernibles of M are special iff $M(\delta)$ is a δ -model, for some (also for every) multiplicatively indecomposable ordinal δ ; and they are remarkable iff in addition the indiscernibles of $M(\delta)$ form a closed cofinal set in $M(\delta)$ (proof in (6) below).

Here are our independent versions of (B): let $\varphi(x_1 \cdots x_n)$ be any open formula in a language not containing the symbol P , but containing $<$; then

(C) $\forall x_1 \cdots x_n \in P$ φ has an ω -model in which P is cofinal $\Leftrightarrow$ it has a finite stretchable model in which P consists of $\geq n$ special indiscernibles.

(D) $\forall x_1 \cdots x_n \in P$ φ has for every uncountable cardinal κ a κ -model in which P is cofinal $\Leftrightarrow$ it has a finite stretchable model as in (C).

(E) $\forall x_1 \cdots x_n \in P$ φ has for every uncountable cardinal κ a κ -model in which P forms a closed cofinal set $\Leftrightarrow$ it has a finite stretchable model in which P consists of $\geq n$ remarkable indiscernibles.

5. THEOREM. (a) *The statement (C) is provable in $\text{ACA}_0 + \text{RT}$ (= first order comprehension + infinite Ramsey theorem), but not in ACA_0 , since $\text{ACA}_0 + (C)$ implies Con ACA_0 .*

(b) *The statement (D) is provable in $\text{ZF} + \forall n < \omega$ (n -Mahlo cardinals exist), but not in ZF since $\text{ZF} + (D)$ implies Con ZF .*

(d) *The statement (E) is provable in $\text{ZF} + (n\text{-ineffable cardinals exist for every } n < \omega)$, but not in $\text{ZF} + (n\text{-Mahlo cardinals exist for every } n < \omega)$, since $\text{ZF} + (E)$ implies $\text{Con}(\text{ZF} + n\text{-Mahlo cardinals})$.*

REMARK. (a) Let S be the set of sentences $\forall \bar{x} \in P \varphi$ which have a finite stretchable model as in (C)–(D); it is easy to see that S is an NP-complete set; thus (C) is the equivalence between an NP condition and a Σ_1^1 condition; and (D) has an “absolute version” which is the equivalence between the the same NP condition and a Π_2^1 one! And similarly with (E). One is tempted to wonder whether such equivalences can be helpful in order to apply ideas coming from descriptive set theory to the study of NP classes....

(b) One can show more precisely under the theory ACA_0 that (C) is equivalent to 1-consistency of Peano’s arithmetic. No such result can hold concerning (D), but at least there may be a theory T , equal to ZF or $\text{ZF} +$ some natural large cardinal axioms, and such that T implies (D) which implies $\text{con } T$. Presently we do not have a really natural T ; the same problem arises for (E).

(c) The “diagram of indiscernibles” of any finite stretchable structure M in which the indiscernibles form a set P is a sentence of the form $\forall \bar{x} \in P \varphi$ which determines M uniquely up to isomorphism. Thus constructing a structure N such that (i) N extends M and (ii) N satisfies a given sentence $\forall \bar{y} \in P \psi$ reduces to ensure that N satisfies $\forall \bar{x}\bar{y} \in P$ (φ and ψ), i.e. to (ii). And although Theorem 5 only talks of realizing (ii), it involves realizing (i). (Thus Theorem 5 fits with the initial idea (0)).

(d) Our original version of Theorem 5.d assumed the existence of a Ramsey cardinal; it is P. Matet who showed us that n -ineffables suffice.

(e) The proof of Theorem 5.d easily implies a stronger consequence of (E): the consistency of $\text{ZF} + S$ (n -Mahlo); where S (n -Mahlo) is the scheme expressing

for every definable application $F: \text{On} \rightarrow \text{On}$ there exists
an n -Mahlo cardinal α such that $F''\alpha = \alpha$.

6. Proof of Fact (4). Assume that X is special in M and Y is an unbounded initial segment of X ; then it is easy to see that $M(Y)$ is an initial segment of X and $M(Y) < x$ for any $x \in X \setminus Y$. Moreover if X is remarkable and $X \setminus Y$ has a least element x , then x is also the least element of $M(X) \setminus M(Y)$. These properties easily imply that when X is a multiplicatively indecomposable ordinal δ , then $M(\delta)$ has the properties required by the Fact.

Conversely assume that $M(\delta)$ is a δ -model for such an ordinal δ ; and that specialness of the set of indiscernibles δ in $M(\delta)$ is violated: $t(\alpha_1 \cdots \alpha_{k+p})$ is less than α_k and differs from $t(\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_p)$ for some $\alpha_1 < \cdots < \alpha_{k+p}$ and $\alpha_k < \beta_1 < \cdots < \beta_p$ in δ . Then by indecomposability of δ there exist more than α_k disjoint sequences $(\beta_1 \cdots \beta_p)$ between α_k and δ ; and by indiscernibility, they all give rise to distinct terms $t(\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_p) < \alpha_k$, contradicting the order type of α_k . A similar argument proves remarkability. Note that the definition of remarkability is inspired by Silver's work on 0^h (see for instance in Chang and Keisler's book *Model theory*, 1973).

7. Proof of (B)–(E), $\Leftarrow$. Assume that M is a stretchable model of $\forall x_1 \cdots x_n \in P \varphi$, and that P^M consists of $\geq n$ indiscernibles; then for every cardinal $\kappa \geq \omega$, its stretching $M(\kappa)$ is a model of $\forall \bar{x} \in P \varphi$ and $P^{M(\kappa)} = \kappa$ is infinite. If in addition P^M was special, then by Fact 4, $M(\kappa)$ is a κ -model, so that $P^{M(\kappa)}$ is cofinal in $M(\kappa)$. If in addition P^M was remarkable, then by Fact 4 $P^{M(\kappa)}$ is closed in $M(\kappa)$. $\square$ (B)–(E), $\Leftarrow$.

PROOF OF (B) AND (E), $\Rightarrow$. Assume that N is a model of $\forall \bar{x} \in P \varphi$ and P^N is infinite; let T_φ denote the set of subterms of the terms occurring in φ , and F_φ denote the set of atomic formulas of the form $R(t_1 \cdots t_n)$, for R in the language of φ (including the equality symbol), and $t_1 \cdots t_n$ in T_φ . Since F_φ is finite, the finite Ramsey theorem implies the existence of a subset X of P^N which is indiscernible with respect to all formulas of F_φ , and is of sufficiently large finite cardinal. Let X_0 be a subset of X such that:

(i) $\text{card } X_0 \geq n$ and $\text{card } X_0 \geq n_\varphi \cdot \max(p, q + 1)$, where n_φ is the largest number of variables of terms of T_φ , and p (resp. q) is the largest number of arguments of relation (resp. function) symbols in the language of φ ,

(ii) before and after element of X_0 , $X \setminus X_0$ contains at least p consecutive elements (in the ordering $<$ of X).

We shall define a finite stretchable weak model of φ by shrinking N to a carefully chosen part of $|N|$ so that the preceding applies, and set:

8. Notation. Let N be any structure with a distinguished element 0; for any subset A of $|N|$,

$N(A)$ denotes the substructure of N generated by A ;

$N|A$ denotes the structure of domain $A \cup \{0\}$ such that for every relation symbol R and function symbol f of the language

$$R^{N|A} = R^N|A \cup \{0\},$$

$f^{N|A}(a_1 \cdots a_n) = f^N(a_1 \cdots a_n)$, except if $a_1 \cdots a_n \in A \cup \{0\}$ are such that

$$f^N(a_1 \cdots a_n) \notin A \cup \{0\},$$

in which case $f^{N|A}(a_1 \cdots a_n) = 0$. Thus $N|A$ is $N(A)$ if A is closed in N , otherwise $N|A$ is a conventional shrinking of $N(A)$ to $A \cup \{0\}$. Finally, we set

$$N(A; T_\varphi) = \{t^N(\bar{a}); \bar{a} \in A^n \varphi, t \in T_\varphi\}.$$

Going back to our model N of $\forall \bar{x} \in P \varphi$, we equip it with a distinguished element 0, so that Notation 8 applies; we set

$$M =_{\text{df}} N[N(X_0) \cap N(X; T_\varphi)] \quad \text{and} \quad P^M =_{\text{df}} X_0.$$

9. Fact. M is a finite model of $\forall \bar{x} \in P \varphi$, and X_0 is a set of indiscernibles generating M .

PROOF OF FACT 9. M is finite because $|M| \subset N(X; T_\varphi)$, and X, T_φ are both finite; that $M \models \forall \bar{x} \in P \varphi$ follows from $|M| \subset N(X; T_\varphi)$ and $P^M \subset X$; and that X_0 generates M follows from $|M| \subseteq N(X_0)$. There remains to see that X_0 keeps its indiscernibility in M . Firstly it is clear that X_0 remains indiscernible with respect to all formulas of F_φ . Using the condition $|M| = N(X_0) \cap N(X; T_\varphi)$, and the choice of $X_0 \subset X$, the proof of point 4.6 of [R1] can be adapted to show that X_0 is fully indiscernible in M . Rather than repeating the proof we indicate its idea:

(i) taking $|M| = N(X_0)$ would most easily have kept the indiscernibility of X_0 but is not possible if $N(X_0)$ is infinite.

(ii) Then the natural idea is to take $|M| = N(X_0; T_\varphi)$ which is finite; but the following could happen: $X_0 = \{x_0, \dots, x_q\}$, $x_0 < \dots < x_{q+1}$ in X , and $N \models 0 \neq f(x_0) = g(x_0 x_1)$, where $f(u) \notin T_\varphi$ and $g(uv) \in T_\varphi$. This guarantees $f^N(x_0) \in M$, hence $f^N(x_0) = f^M(x_0) \neq 0$; on the other hand, since $f \in T_\varphi$ and $x_{q+1} \in X \setminus X_0$ it might be that $f^N(x_q) = g^N(x_q x_{q+1})$ is not in $N(X_0; T_\varphi)$, i.e. in M under our present hypothesis. Then $M \models 0 = f(x_q)$, contradicting the indiscernibility of X_0 in M .

(iii) Choosing M and X_0 as we did avoids the trouble of (ii) (for one can argue as follows: $N \models 0 \neq f(x_0) = g(x_0 x_1)$ implies $N \models 0 \neq f(x_q) = g(x_q x_{q+1})$ by indiscernibility of X_0 in N . Then $f^N(x_q) \in N(X_0)$, $g^N(x_q x_{q+1}) \in N(X; T_\varphi)$, so $f^N(x_q) = g^N(x_q x_{q+1}) \in N(x_0) \cap N(X; T_\varphi) = |M|$, hence $f^M(x_q) = f^N(x_q) \neq 0$ as it should). $\square$ Fact 9.

It is easy to expand M with new functions so that in this expansion X_0 remains indiscernible and every element b of M becomes of the form $b = t^M(\bar{x})$, for some term t and some $\bar{x} \in X_0^{\leq n_\varphi}$ (indeed, b is already of the form $t^M(\bar{y})$ with $t \in T_\varphi$ and $\bar{y} \in X^{\leq n_\varphi}$; let $\bar{x}$ be the restriction to X_0 of $\bar{y}$, introduce a new function symbol f with arity $k =_{\text{df}}$ length of $\bar{x}$, and set $f^M(\bar{x}) =_{\text{df}} b$, $f^M(\bar{c}) =_{\text{df}} 0$ for every $\bar{c}$ in $M^k \setminus X_0^k$). Then since X_0 was of power $\geq n_\varphi \cdot \max(p, q + 1)$, M is a stretchable structure; this shows (B), $\Rightarrow$. In order to prove (C), $\Rightarrow$, consider the above proof, when the given structure N is an ω -model; we only have to ensure the additional property that X_0 is special in M . The infinite Ramsey theorem provides an infinite subset Y of P^N which is indiscernible with respect to the formulas of F_φ ; where F_φ now has $<$ among its relation symbols. Then Y is of type ω , and

10. Fact. Y is special with respect to all terms of T_φ i.e. for all $s(v_1 \cdots v_n)$, $t(v_1 \cdots v_{n+k})$ in T_φ and all $x_1 < \dots < x_{n+k}$ in Y , N satisfies

(i) $s(x_1 \cdots x_n) < x_{n+1}$,

(ii) $[t(x_1 \cdots x_{n+k}) < x_n \rightarrow t(x_1 \cdots x_{n+k}) = t(x_1 \cdots x_n y_1 \cdots y_k)]$ whenever $x_{n+1} < y_1 < \dots < y_k$ in Y .

PROOF OF FACT 10. If (i) did not hold, by indiscernibility $s(x_1 \cdots x_n)$ would be greater than x_q for all $q > n$, contradicting the order type ω of N . If (ii) did not hold, indiscernibility and the infinite character of Y would allow to choose an arbitrary large finite number of terms $t(x_1 \cdots x_n y^i)$ with $y^i \in Y^k$, all distinct and smaller than x_{n+1} ; which would again be in contradiction with the order type ω of N . $\square$ Fact 10.

Now choose a subset X of Y of sufficiently large finite power, and construct X_0 , M from X , N as in the preceding proof; $X_0 \subset Y$ easily implies that X_0 is special in M , showing (C), $\Rightarrow$.

PROOF OF (D), $\Rightarrow$. As indicated in Theorem 5.b, we assume the existence of Mahlo cardinals of every finite order; consider the preceding proof, when N is assumed to be a κ -model, where κ is a Mahlo cardinal of large finite order. In such a case we can find the set Y of this proof by applying a partition property of κ instead of the infinite Ramsey theory, and thus obtain a much larger set Y ; more precisely we shall apply

11.

THEOREM. $\forall n < \omega \exists p(n) < \omega$ such that every $p(n)$ -Mahlo cardinal κ satisfies: for every partition $f: [\kappa]^n \rightarrow n$ and every function $g: \kappa \rightarrow \kappa$, there exists a homogeneous set Y of cardinal $> g(\alpha)$, where α is the n th element of Y .

PROOF. Similar to the proof of Theorem 3.11 of [Mc-R]. $\square$ Theorem 11.

In order to get the set $Y \subset P^N$, we fix $N_\varphi < \omega$ and a partition $f: [\kappa]^{N_\varphi} \rightarrow N_\varphi$ such that every homogeneous set will be indiscernible with respect to F_φ —when F_φ is defined as before. We can assume that $|N| = \kappa$, and that κ is p_φ -Mahlo, when p_φ is the $p(n)$ provided by Theorem 11 for $n = N_\varphi$; we then obtain $Y \subset P^N$, indiscernible with respect to F_φ and of cardinal $> g(\alpha)$, where α is the N_φ th element of Y , and g is any given function: $\kappa \rightarrow \kappa$ —we choose $g(\alpha) =_{\text{df}} \sup_{\bar{x} < \alpha, t \in T_\varphi} t^N(\bar{x})$.

12. Fact. Y is special with respect to T_φ .

PROOF OF FACT 12. This is a copy of the proof of Fact 10, with the following changes: by indiscernibility of Y we can assume that $x_1 \cdots x_n$ are the n first elements of Y , so that $x_n = \alpha$, so $\text{card } Y > g(x_n)$. Then if (i) failed, the next $g(x_n) - n$ elements of Y after x_n would be $< s(x_1 \cdots x_n)$ by indiscernibility, contradicting $s(x_1 \cdots x_n) < g(x_n)$. If (ii) failed, a similar modification of the previous proof, done with a somewhat larger function g , would do. $\square$ Fact 12.

From Y, N we construct X_0, M as already done in the proof of (C), $\Rightarrow$. $\square$ (D), $\Rightarrow$.

PROOF OF (E), $\Rightarrow$. Assume that in the previous proof κ was a p -ineffable cardinal; then by Proposition 4.3.ii of [M], given any functions $f_i: [\kappa]^{k+p} \rightarrow \kappa$, $i < 1$, there exists a stationary set $B \subseteq \kappa$ such that B is remarkable with respect to the f_i 's. We apply this when the functions f_i are the interpretation in N of all terms of T_φ . Then since B is stationary $B \cap P$ is unbounded in κ , and taking the set Y of the proof to be $B \cap P$ ensures that Y will be remarkable with respect to T_φ . So that $X_0 \subset X \subset Y$ will imply that X_0 is remarkable, in N as well as M . $\square$ (E), $\Rightarrow$.

Thus we proved Theorem 5 except for its independence results: (C) $\Rightarrow$ Con ACA_0 etc.; they depend on another independence result, Theorem 4.II of [R1], which provides three sets C_i , $i < 2$, of sentences of the form $\forall \bar{x} \in P \varphi$, such that the existence

of finite stretchable models M for the sentences of C_i implies the consistency of ACA_0 , ZF and $\text{ZF} + \text{Mahlo}$ cardinals, respectively. The present proof of Theorem 5 will consist in obtaining these finite models M required by [R1, Theorem 4.II]; to that end, we construct for each cardinal $\kappa \geq \omega$ a κ -model N . These models N will satisfy the hypotheses of (C), (D), (E), so that by applying part $\Rightarrow$ of (C), (D), (E) we shall obtain the required finite models M .

For each ordinal α , L_α denotes the collection of all sets constructible at stage $\leq \alpha$, and $<$ denotes the constructible ordering on L_α . For each cardinal κ , L_κ is closed under the eight following operations on sets a, b : pair $\{a, b\}$, difference $a \setminus b$, cartesian product $a \times b$, first projection Πa , graph $\varepsilon|a$, permutations αa , βa (defined by $\alpha a =_{\text{df}} \{((x, y), z); (x, (y, z)) \in a\}$ and $\beta a =_{\text{df}} \{((x, z), y); (x, (y, z)) \in a\}$), and minimum $\Box a$ ($=_{\text{df}}$ smallest element of a in the constructible order on L). Thus if L_{BST} is the language consisting of ε , $<$ together with the eight above function symbols; and if BST denotes extensionality plus the eight axioms defining the eight above operations, then L_κ has a natural expansion to L_{BST} which is a model of BST, and which we still denote L_κ . We let L be L_{BST} together with the predicate symbol P , and we define the L -structure N so that

$$N|_{L_{\text{BST}}} = L_\kappa,$$

$$P^N = \begin{cases} \{L_n; n < \omega\} & \text{if } \kappa = \omega, \\ \{L_\delta; \delta \text{ limit ordinal } < \kappa\} & \text{if } \kappa > \omega. \end{cases}$$

Then N satisfies

(i) $\forall \bar{x} \in P \varphi$, for every open formula φ of L_{BST} such that $\forall \bar{x} \varphi$ is a consequence of BST.

(ii) $\forall x \forall y (x \in y \rightarrow x < y) \forall x \forall y [P(y) \rightarrow (x < y \leftrightarrow x \in y)]$.

And if $\kappa > \omega$, then N satisfies for all terms $t(\bar{x})$ of L_{BST} .

(iii) $\forall \bar{x} \in P [t(\bar{x}) \in x_1 \rightarrow \{t(\bar{x})\} \in x_1]$, (where x_1 is the first coordinate of $\bar{x}$).

Assume (C); since (i) + (ii) has an ω -model in which P is cofinal, by (C) every finite part of (i) + (ii) has a finite model M , in which P consists of n special indiscernibles, for any $n < \omega$. From (ii) follows that special indiscernibles in the present sense remain special in the sense of [R1], that is when ε replaces $<$, in the condition (i + ii) of Definition 3. Thus we proved that every finite part of (i) has a model in which P consists of n special indiscernibles, and by Theorem 4.II.a of [R1] this is equivalent to Con ACA_0 . $\square$ Theorem 5.b for (C).

Assume (D); since for every $\kappa > \omega$ (i) + (ii) + (iii) has a κ -model in which P is cofinal, by (D) every finite part of (i) + (ii) + (iii) has a model in which P consists of n special indiscernibles—both in the sense of Definition 3 and of [R1]. By Theorem 4.II.b of [R1] this implies Con ZF . $\square$ Theorem 5.b for (D).

Assume (E); since for every $\kappa > \omega$ (i) + (ii) has a κ -model in which P is closed cofinal, by (E) every finite part of (i) + (ii) has a model in which P is a set of n remarkable indiscernibles—in the sense of Definition 3, but (using (ii)) also in the sense of [R1]. Then Theorem 4.II.c of [R1] implies $\text{Con}(\text{ZF} + n\text{-Mahlo cardinals})$, for each integer n . $\square$ Theorem 5.

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Nota bene: the independence results of Theorem 5 rest on a study of nonstandard models of Set theory, part of which can be found only in [R], which appears in [R-W]. And [R-W] can be obtained by sending an order to "Publications mathématiques de l'Université Paris 7" at the address below (payment after reception of the volume, price 95 F+ port. In addition to [R], the volume contains a course due to A. Wilkie on nonstandard models of arithmetic and their application to algorithmic complexity.)

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