A GENERALIZED CONVERSE MEASURABILITY THEOREM

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ABSTRACT. It has long been known that measurability questions in Wiener space and Yeh-Wiener space are often rather delicate. Some converse measurability theorems in Wiener and Yeh-Wiener spaces were proved by Köehler, Skoug, and the first author.

In this paper, we establish a generalized converse measurability theorem by which the above measurability theorems are proved as corollaries.

1. Introduction. Let $C_1[a, b]$ denote the Wiener space of functions of one variable i.e., $C_1[a, b] = \{x(\cdot) | x(a) = 0 \text{ and } x(s) \text{ is continuous on } [a, b]\}$. Let $R = \{(s, t) | a \leq s \leq b, \alpha \leq t \leq \beta\}$ and let $C_2[R]$ be the Yeh-Wiener space (or 2-parameter Wiener space), i.e., $C_2[R] = \{x(\cdot, \cdot) | x(a, t) = x(s, \alpha) = 0, x(s, t) \text{ is continuous on } R\}$. Let ν be Wiener measure on $C_1[a, b]$ and let λ be Yeh-Wiener measure on $C_2[R]$. For a discussion of Yeh-Wiener measure see [1, 5, and 6]. Note that Wiener space and Yeh-Wiener space are separable Banach spaces with respect to the supremum norm.

Let $a = t_0 < t_1 < \cdots < t_n = b$ be a subdivision of $[a, b]$. Let E be any subset of Euclidean space $\mathbf{R}^n$ and define $J: C_1[a, b] \rightarrow \mathbf{R}^n$ by

$$(1) \quad J(x) = (x(t_1), x(t_2), \dots, x(t_n)).$$

J is continuous on $C_1[a, b]$ with respect to the uniform topology. By the definition of Wiener measure, if E is Lebesgue measurable then $J^{-1}(E)$ is Wiener measurable and

$$(2) \quad \nu(J^{-1}E) = \int_E g(\vec{\xi}) d\gamma(\vec{\xi})$$

where γ is Lebesgue measure on $\mathbf{R}^n$,

$$g(\vec{\xi}) = [(2\pi)^n (t_1 - a)(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-1/2} \\ \times \exp \left\{ -\frac{1}{2} \sum_{i=1}^n (\xi_i - \xi_{i-1})^2 / (t_i - t_{i-1}) \right\} > 0,$$

and $\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n$.

In an unpublished result Köehler established the converse, i.e., if $J^{-1}(E)$ is Wiener measurable then E is Lebesgue measurable. In [5], Skoug extended Köh-

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ler's result to Yeh-Wiener space and established the converse of the one line theorem in [1]. In [2], the first author extended Skoug's result to the n -parallel lines theorem in [1]. Later the proof of the converse of the n -parallel lines theorem was simplified [3].

In this paper, we establish a generalized converse measurability theorem which has the above converse measurability theorems as corollaries. The techniques used in this paper are quite different from the techniques used in [2, 3, and 5].

2. A generalized converse measurability theorem. A probability measure P on a σ -algebra $\mathcal{S}$ containing the Borel sets in a topological space is called tight if for any $\varepsilon > 0$ and for any E in $\mathcal{S}$ there exists a compact set $K \subset E$ such that $P(E \setminus K) < \varepsilon$. It is known that any probability measure on the Borel class of a complete separable metric space is tight [4].

Let P be a tight measure on $\mathcal{B}(S)$ and let m be a measure on $\mathcal{B}(T)$ where $\mathcal{B}(S)$ and $\mathcal{B}(T)$ are the Borel classes of topological spaces S and T , respectively. Let $(S, \mathcal{B}(S), \bar{P})$ and $(T, \mathcal{B}(T), \bar{m})$ be the completions of $(S, \mathcal{B}(S), P)$ and $(T, \mathcal{B}(T), m)$, respectively. It is easy to see that $\bar{P}$ is tight on $\mathcal{B}(S)$.

Let $J: S \rightarrow T$ be continuous and let $\mathcal{U} = \{E \subset T: J^{-1}(E) \text{ is } \bar{P}\text{-measurable}\}$.

Define a set function μ on $\mathcal{U}$ by $\mu E = \bar{P}(J^{-1}(E))$ where $E \in \mathcal{U}$. $\mathcal{U}$ is a σ -algebra which contains the Borel sets in T as is easily checked and $(T, \mathcal{U}, \mu)$ is a complete tight measure space. To see that μ is tight on $\mathcal{U}$, let E be μ -measurable. Then for each positive integer n , there exists a compact subset K_n of $J^{-1}(E)$ such that $\bar{P}(J^{-1}(E) \setminus K_n) < 1/n$. Put $C_n = \bigcup_{i=1}^n K_i$. Then $\bar{P}(J^{-1}(E) \setminus C_n) < 1/n$. Since C_n is compact, $J(C_n)$ is compact and

$$\mu(E \setminus J(C_n)) = \bar{P}(J^{-1}(E \setminus J(C_n))) \leq \bar{P}(J^{-1}(E) \setminus C_n) < 1/n.$$

Hence μ is tight on $\mathcal{U}$.

LEMMA 2.1. $\mathcal{U} = \overline{\mathcal{B}(T)}$ under the following assumption: N is an $\bar{m}$ -null set if and only if N is a μ -null set.

PROOF. It is easy to see that $\overline{\mathcal{B}(T)} \subset \mathcal{U}$. To show that $\mathcal{U} \subseteq \overline{\mathcal{B}(T)}$, let $E \in \mathcal{U}$. Since μ is tight, there exists a compact set K_n such that $K_n \subset E$ and $\mu(E \setminus K_n) < 1/n$ for each $n = 1, 2, \dots$. Let $K = \bigcup_{n=1}^{\infty} K_n$. Then K is a Borel set and $K \subset E$. Since $\mu(E \setminus K) \leq \mu(E \setminus K_n) < 1/n$ for each $n = 1, 2, \dots$, $\mu(E \setminus K) = 0$. By assumption, $E \setminus K$ is an $\bar{m}$ -null set. Then $E = K \cup (E \setminus K)$ is $\bar{m}$ -measurable and so $\mathcal{U} \subset \overline{\mathcal{B}(T)}$.

For the measurability theorems in Wiener and Yeh-Wiener spaces in which we are concerned, we have $\overline{\mathcal{B}(T)} \subset \mathcal{U}$. For example, if E is Lebesgue measurable then $J^{-1}(E)$ is Wiener measurable. To establish the converse measurability theorems we need $\mathcal{U} = \overline{\mathcal{B}(T)}$.

THEOREM 2.2. Suppose $\overline{\mathcal{B}(T)} \subset \mathcal{U}$. If there exists an integrable function $g: T \rightarrow R$ such that $g(t) > 0$ for $\bar{m}$ -a.e. t and $\mu E = \int_E g(t) d\bar{m}(t)$ for every $\bar{m}$ -measurable set E , then $\mathcal{U} = \overline{\mathcal{B}(T)}$.

PROOF. By Lemma 2.1, it suffices to show that if N is a μ -null set then N is an $\bar{m}$ -null set. Now we assume that N is a μ -null set and it is not an $\bar{m}$ -null set. We consider two cases: (1) N is $\bar{m}$ -measurable, (2) N is not $\bar{m}$ -measurable.

If N is $\overline{m}$ -measurable, then there exists a Borel set $B \subset N$ such that $mB = \overline{m}N > 0$. Since $\mu B = \int_B g(t) d\overline{m}(t) > 0$, $\mu_* N > 0$ where μ_* is the inner measure of μ . Since $\mu N = \mu_* N$, $\mu N > 0$ which is a contradiction.

If N is not $\overline{m}$ -measurable, i.e., $N \notin \overline{\mathcal{B}}(T)$, then $mB > 0$ for every Borel set $B \supset N$. Then $\mu B > 0$ for any Borel set $B \supset N$. Since μ is tight, there exists a compact set $K_n \subset N^c$ such that $\mu(N^c \setminus K_n) < 1/n$ for each n . Let $K = \bigcup_{n=1}^{\infty} K_n$. Then K^c is a Borel set and $N \subset K^c$.

$$\begin{aligned}\mu(K^c) &= 1 - \mu(K) = \mu(N^c) - \mu(K) \leq \mu(N^c \setminus K) \\ &\leq \mu(N^c \setminus K_n) < 1/n \quad \text{for each } n.\end{aligned}$$

Hence $\mu(K^c) = 0$. This contradicts the fact that $\mu B > 0$ for any Borel set $B \supset N$. Therefore we conclude that every μ -null set is also an $\overline{m}$ -null set.

3. Converse measurability theorems.

THEOREM 3.1 (KÖEHLER). *Let $a = t_0 < t_1 < \dots < t_n = b$. Let E be any subset of $\mathbf{R}^n$ and let J be defined as in (1). Then E is Lebesgue measurable if and only if $J^{-1}(E)$ is Wiener measurable.*

PROOF. By the definition of Wiener measure, if E is Lebesgue measurable, then $J^{-1}(E)$ is Wiener measurable. To show the converse, let

$$\mathcal{U} = \{E \subset \mathbf{R}^n | J^{-1}(E) \text{ is Wiener measurable}\}.$$

Then $\mathcal{U}$ contains all Lebesgue measurable sets in $\mathbf{R}^n$. For $E \in \mathcal{U}$, we define μ on $\mathcal{U}$ by $\mu E = \nu(J^{-1}(E))$. Then every Lebesgue measurable set E satisfies (2). Since Wiener measure is tight on the σ -algebra of Wiener measurable sets which contains the Borel sets in $C_1[a, b]$, $(\mathbf{R}^n, \mathcal{U}, \mu)$ is a complete tight measure space. Hence by Theorem 2.2, $J^{-1}(E)$ is Wiener measurable if and only if E is Lebesgue measurable.

Let $a = s_0 < s_1 < \dots < s_m = b$, $\alpha = t_0 < t_1 < \dots < t_n = \beta$. Let E be any subset of $\mathbf{R}^{mn}$ and define $J: C_2[R] \rightarrow \mathbf{R}^{mn}$ by

$$(3) \quad J(x) = (x(s_1, t_1), \dots, x(s_m, t_n)).$$

J is continuous on $C_2[R]$ with respect to the uniform topology. By the definition of Yeh-Wiener measure, if E is Lebesgue measurable then $J^{-1}(E)$ is Yeh-Wiener measurable and

$$(4) \quad \lambda(J^{-1}(E)) = \int_E g(\vec{\xi}) d\gamma(\vec{\xi})$$

for any Lebesgue measurable set E in $\mathbf{R}^{mn}$, where γ is Lebesgue measure on $\mathbf{R}^{mn}$,

$$\begin{aligned}g(\vec{\xi}) &= \{(2\pi)^{mn} \{(s_1 - a) \cdots (s_m - s_{m-1})(t_1 - \alpha) \cdots (t_n - t_{n-1})\}^{-1/2} \\ &\times \exp \left\{ -\frac{1}{2} \sum_{k=1}^n \sum_{j=1}^m (\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2 / (s_j - s_{j-1})(t_k - t_{k-1}) \right\} \\ &> 0,\end{aligned}$$

and $\vec{\xi} = (\xi_{1,1}, \xi_{1,2}, \dots, \xi_{m,n}) \in \mathbf{R}^{mn}$.

We will omit the proof of Theorem 3.2 below since it is similar to the proof given above for Theorem 3.1.

THEOREM 3.2 (SKOUG). *Let J be defined as in (3) and let E be any subset of $\mathbb{R}^{mn}$. Then E is Lebesgue measurable if and only if $J^{-1}(E)$ is Yeh-Wiener measurable.*

We shall use the following notation for the Cartesian product of n Wiener spaces $\overset{n}{X}C_1[a, b] = C_1[a, b] \times \cdots \times C_1[a, b]$ and $\overset{n}{X}\nu = \nu \times \cdots \times \nu$ will denote the product of n Wiener measures on $\overset{n}{X}C_1[a, b]$.

Let $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ be a subdivision of $[\alpha, \beta]$. Define $\psi: \overset{n}{X}C_1[a, b] \rightarrow \overset{n}{X}C_1[a, b]$ by

$$\psi(y_1, y_2, \dots, y_n) = \left(\sqrt{\frac{t_1 - t_0}{2}} y_1, \sqrt{\frac{t_1 - t_0}{2}} y_1 + \sqrt{\frac{t_2 - t_1}{2}} y_2, \dots, \right. \\ \left. \sqrt{\frac{t_1 - t_0}{2}} y_1 + \cdots + \sqrt{\frac{t_n - t_{n-1}}{2}} y_n \right).$$

Then ψ is 1-1 and onto. ψ and ψ^{-1} are continuous with respect to the uniform topology. Let $G: C_2[R] \rightarrow \overset{n}{X}C_1[a, b]$ be defined by $G(x) = (x(\cdot, t_1), \dots, x(\cdot, t_n))$. Then G is a continuous function from $C_2[R]$ onto $\overset{n}{X}C_1[a, b]$. In [1] Cameron and Storvick evaluated certain Yeh-Wiener integrals in terms of Wiener integrals. In particular they obtained the n -parallel lines theorem. The converse of the n -parallel lines theorem follows quite easily once Theorem 3.3 below is established [2].

THEOREM 3.3. *Let A be any subset of $\overset{n}{X}C_1[a, b]$. Then $\psi^{-1}A$ is $\overset{n}{X}\nu$ -measurable if and only if $G^{-1}A$ is Yeh-Wiener measurable. Furthermore,*

$$\lambda(G^{-1}A) = \overset{n}{X}\nu(\psi^{-1}A).$$

PROOF. By the n -parallel lines theorem, if $\psi^{-1}A$ is $\overset{n}{X}\nu$ -measurable then $G^{-1}A$ is Yeh-Wiener measurable and $\lambda(G^{-1}A) = \overset{n}{X}\nu(\psi^{-1}A)$ [2]. To show the converse, let $J = \psi^{-1} \circ G$. Then J is a continuous function from $C_2[R]$ onto $\overset{n}{X}C_1[a, b]$. Let

$$\mathcal{U} = \{E \in \overset{n}{X}C_1[a, b]: J^{-1}(E) \text{ is Yeh-Wiener measurable}\}.$$

Define a set function μ on $\mathcal{U}$ by $\mu E = \lambda(J^{-1}E)$. For any $\overset{n}{X}\nu$ -measurable set $E = \psi^{-1}(\psi E)$, $G^{-1}(\psi E) = J^{-1}(E)$ is Yeh-Wiener measurable and $\lambda(G^{-1}\psi E) = \overset{n}{X}\nu(\psi^{-1}\psi E) = \overset{n}{X}\nu(E)$ by the n -parallel lines theorem. Hence $E \in \mathcal{U}$ and $\mu E = \lambda(J^{-1}E) = \lambda(G^{-1}\psi E) = \overset{n}{X}\nu(E) = \int_E 1 d\overset{n}{X}\nu$. Hence by Theorem 2.2, $\mathcal{U}$ is the σ -algebra of $\overset{n}{X}\nu$ -measurable sets. If $G^{-1}A = J^{-1}(\psi^{-1}A)$ is Yeh-Wiener measurable, then $\psi^{-1}A \in \mathcal{U}$ and so $\psi^{-1}A$ is $\overset{n}{X}\nu$ -measurable.

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REFERENCES

1. R. H. Cameron and D. A. Storvick, *Two related integrals over spaces of continuous functions*, Pacific J. Math. **55** (1974), 19–37.
2. K. S. Chang, *Converse measurability theorems for Yeh-Wiener space*, Pacific J. Math. **97** (1981), 59–63.
3. K. S. Chang and I. Yoo, *A simple proof of converse measurability theorem for Yeh-Wiener spaces*, Bull. Korean Math. Soc. **23** (1986), 35–37.
4. K. R. Parthasarathy, *Probability measures on metric space*, Academic Press, New York, 1967.
5. D. L. Skoug, *Converses to measurability theorems for Yeh-Wiener space*, Proc. Amer. Math. Soc. **57** (1976), 304–310.
6. J. Yeh, *Wiener measure in a space of functions of two variables*, Trans. Amer. Math. Soc. **95** (1960), 433–450.

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