PSEUDOPRIME l -IDEALS IN A CLASS OF f -RINGS

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ABSTRACT. In a commutative f -ring, an l -ideal I is called pseudoprime if $ab = 0$ implies $a \in I$ or $b \in I$, and is called square dominated if for every $a \in I$, $|a| \leq x^2$ for some $x \in A$ such that $x^2 \in I$. Several characterizations of pseudoprime l -ideals are given in the class of commutative semiprime f -rings in which minimal prime l -ideals are square dominated. It is shown that the hypothesis imposed on the f -rings, that minimal prime l -ideals are square dominated, cannot be omitted or generalized.

Introduction. Let X be a topological space and $C(X)$ be the f -ring of all continuous real-valued functions on X with coordinatewise operations. The following characterizations of pseudoprime l -ideals of $C(X)$ are known.

(L. Gillman and C. Kohls [4, 4.1]) *For an l -ideal I of $C(X)$, the following are equivalent:*

- (1) *I is pseudoprime.*
- (2) *The prime ideals containing I form a chain.*
- (3) *$\sqrt{I}$ is prime.*

In [11], Subramanian asks whether this characterization of pseudoprime l -ideals generalizes to semiprime f -rings. The answer in general is no, as can be seen by Example 2.7. In this work, we investigate pseudoprime l -ideals in the class of commutative semiprime f -rings in which minimal prime l -ideals are square dominated. In this class of f -rings, we give some alternate characterizations of pseudoprime l -ideals, and we show that in normal f -rings conditions (2) and (3) characterize pseudoprime l -ideals. We also show that if all prime l -ideals are square dominated, a generalization of condition (3) characterizes pseudoprime l -ideals in archimedean f -algebras. Finally, we show that the hypothesis imposed on our f -rings, that minimal prime l -ideals be square dominated, cannot be omitted or generalized in any way by showing that if any of the characterizations hold in a semiprime f -ring A , then all minimal prime l -ideals of A are square dominated. We assume throughout that all rings are commutative and semiprime.

1. Preliminaries. An f -ring is a lattice ordered ring which is a subdirect product of totally ordered rings. For general information on f -rings see [2]. Given an f -ring A and $x \in A$, we let $A^+ = \{a \in A: a \geq 0\}$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$.

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A ring ideal I of an f -ring A is an l -ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any subset $S \subset A$, there is a smallest l -ideal containing S , and we denote this by $\langle S \rangle$.

Suppose A is an f -ring and I, J are l -ideals of A . We let $I : J = \{a \in A : aJ \subseteq I\}$. The l -ideal I is semiprime (prime) if $a^2 \in I$ ($ab \in I$) implies $a \in I$ ($a \in I$ or $b \in I$). The f -ring A is semiprime (prime) if $\{0\}$ is a semiprime (prime) l -ideal. We let $\sqrt{I}$ denote $\{a \in A : a^n \in I \text{ for some } n\}$, the smallest semiprime l -ideal containing I . In [5, 3.5], it is shown that:

(1.1) If I is an l -ideal of a semiprime f -ring A , then $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is semiprime.

It is well known that all l -ideals containing a given prime l -ideal form a chain. The following result is also well known.

(1.2) A prime l -ideal P of a commutative semiprime f -ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$.

A subset M of an f -ring A is called an m -system if whenever $a, b \in M$ there exists an $x \in A$ such that $axb \in M$. If in A , there is an l -ideal I and an m -system M such that $I \cap M = \emptyset$, then there is a prime l -ideal P such that $I \subset P$, and $P \cap M = \emptyset$.

We call an ideal I *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$. In a semiprime f -ring A , a pseudoprime l -ideal contains a prime l -ideal as shown in [11, 2.1]. Also, in a commutative f -ring a pseudoprime and semiprime l -ideal is necessarily a prime l -ideal as shown in [8, 4.2] and [11, 2.5].

An ideal I of a commutative f -ring A with identity element is a z -ideal if whenever $a, b \in A$, are contained in the same set of maximal ideals and $a \in I$ then $b \in I$. In [10, 2.7], G. Mason shows

(1.3) In a commutative f -ring A , a z -ideal I is prime if and only if for all $a \in A$, either $a^+ \in I$ or $a^- \in I$.

Henriksen, in [5] calls an l -ideal I of an f -ring A square dominated if $I = \{a \in A : |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}$. Every prime square dominated l -ideal P satisfies $P = \langle P^2 \rangle$.

An f -ring (and more generally a Riesz space) A is called normal if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. Several conditions equivalent to normality for an f -ring can be found in [8, 6.3], and for a Riesz space in [7, Theorem 9].

Given an f -ring A and an element $x > 0$ in A , the sequence $\{f_n\}_{n=1}^{\infty}$ is said to *converge x -uniformly* to the element $f \in A$ if for every $\varepsilon > 0$, there exists a positive integer N_ε such that $|f - f_n| < \varepsilon x$ for all $n \geq N_\varepsilon$. An x -uniform Cauchy sequence is defined similarly. If for every $x \geq 0$, every x -uniform Cauchy sequence has a unique limit, then A is said to be *uniformly complete*. We say A is *archimedean* if $a, b \in A^+$ with $na \leq b$ for all n implies $a = 0$. If A is an archimedean f -algebra with identity element, it is well known that A is commutative and semiprime. In [1, 4.1(d)], it is shown that:

(1.4) For any archimedean f -algebra A with identity element, there is an embedding e of A into a uniformly complete f -algebra A^* (the uniform completion of A).

2. In this section we give several characterizations of pseudoprime l -ideals in the class of commutative semiprime f -rings with minimal prime l -ideals square dominated. This class contains, of course, all commutative semiprime square root

closed f -rings. First we give a lemma that will be used later and that also gives a characterization of commutative semiprime f -rings in which minimal prime l -ideals are square dominated.

LEMMA 2.1. *Let A be a commutative semiprime f -ring.*

(1) *A semiprime l -ideal I is square dominated if every prime l -ideal minimal with respect to containing I is square dominated.*

(2) *Every minimal prime l -ideal of A is square dominated if and only if for every $a \in A^+$, the l -ideal $\{a\}^d = \{b \in A : ab = 0\}$ is square dominated.*

PROOF. (1) Let $b \in I^+$. Let $M = \{c_1^2 \cdots c_n^2 : n \in \mathbb{N}; b \leq c_i^2\}$. Then M is an m -system. Suppose that $M \cap I = \emptyset$. Then there is a prime l -ideal P such that $I \subseteq P$ and $M \cap P = \emptyset$. Let $P_1 \subseteq P$ be a prime l -ideal minimal with respect to containing I . By hypothesis, P_1 is square dominated, so there is a $p \in A$ such that $b \leq p^2$ and $p^2 \in P_1$. But then $p^2 \in M \cap P$, contrary to assumption. So $M \cap I \neq \emptyset$. Let $c_1^2 \cdots c_n^2 \in M \cap I$, where $b \leq c_i^2$ for each i . Then $b \leq c_1^2 \wedge \cdots \wedge c_n^2 = (|c_1| \wedge \cdots \wedge |c_n|)^2$. Also, $0 \leq (|c_1| \wedge \cdots \wedge |c_n|)^{2n} \leq c_1^2 \cdots c_n^2$. This implies $(|c_1| \wedge \cdots \wedge |c_n|)^{2n} \in I$, and because I is semiprime, $(|c_1| \wedge \cdots \wedge |c_n|)^2 \in I$.

(2) $\Rightarrow$ Suppose that every minimal prime l -ideal is square dominated. Let $a \in A^+$. Suppose P is a prime l -ideal minimal with respect to containing $\{a\}^d$. Then $M = \{b : b \in A \setminus P\} \cup \{a^n : n \in \mathbb{N}\} \cup \{ba^n : b \in A \setminus P, n \in \mathbb{N}\}$ is an m -system such that $M \cap \{a\}^d = \emptyset$. So there is a prime l -ideal P_1 satisfying $\{a\}^d \subseteq P_1 \subseteq P$. But our choice of P implies $P_1 = P$ and $a \notin P$.

Now if P_2 is a minimal prime l -ideal contained in P , $a \notin P_2$ implies $\{a\}^d \subseteq P_2$. Hence $P_2 = P$, and P is in fact a minimal prime l -ideal which is square dominated. So every prime l -ideal minimal with respect to containing $\{a\}^d$ is square dominated, and part (1) implies $\{a\}^d$ is square dominated.

$\Leftarrow$ Let P be a minimal prime l -ideal, and $f \in P$. By 1.2, there is a $g \notin P$ such that $fg = 0$. By hypothesis, $\{g\}^d$ is square dominated. So there is $f_1 \in \{g\}^d$ such that $f \leq f_1^2$ and $f_1^2 \in P$. $\square$

Our first characterization of pseudoprime l -ideals follows.

THEOREM 2.2. *Let A be a commutative, semiprime f -ring with identity element and in which every minimal prime l -ideal is square dominated. The following are equivalent for an l -ideal I :*

- (1) *I is pseudoprime.*
- (2) *$\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is prime.*
- (3) *$\langle I\sqrt{I} \rangle$ is pseudoprime.*
- (4) *$I : \sqrt{I}$ is pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is prime.*

PROOF. (1) $\Rightarrow$ (2). Let P be a minimal prime l -ideal contained in I . Since P is square dominated, $P = \bigcap_{n=1}^{\infty} \langle P^n \rangle \subseteq \bigcap_{n=1}^{\infty} \langle I^n \rangle$. So $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is pseudoprime. By 1.1, it is also semiprime and therefore prime.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). By (3), all l -ideals containing $\langle I\sqrt{I} \rangle$ form a chain. So $I : \sqrt{I} \subseteq \sqrt{I}$ or $\sqrt{I} \subseteq I : \sqrt{I}$.

(4) $\Rightarrow$ (1). Either hypothesis implies that there is a minimal prime l -ideal P contained in $\sqrt{I}$ and also in $I : \sqrt{I}$. Then since P is square dominated, $P = \langle P^2 \rangle \subseteq (I : \sqrt{I})\sqrt{I} \subseteq I$. $\square$

Recall that in a commutative ring, an ideal I is *primary* if $ab \in I$, $a \notin I$ implies $b^n \in I$ for some n . It is well known that $\sqrt{I}$ is prime for every primary l -ideal I . Thus in $C(X)$, every primary l -ideal is pseudoprime. This is not true in general (as can be seen by Example 2.7) and the following corollary gives a condition under which primary l -ideals are pseudoprime.

COROLLARY 2.3. *Let A be a commutative semiprime f -ring with identity element in which every minimal prime l -ideal is square dominated. A primary l -ideal I is pseudoprime if and only if $I : \sqrt{I}$ is pseudoprime.*

PROOF. Assume that $I : \sqrt{I}$ is pseudoprime. If $I = \sqrt{I}$, then I is prime, so we may assume that $I \neq \sqrt{I}$. Let $a \in \sqrt{I} \setminus I$. We will show $I : \sqrt{I} \subseteq \sqrt{I}$. Suppose $b \in I : \sqrt{I}$. Then $ab \in I$, and since $a \notin I$, we must have $b \in \sqrt{I}$. So $I : \sqrt{I} \subseteq \sqrt{I}$. It follows from the previous theorem that I is pseudoprime. $\square$

It is well known that in $C(X)$, an l -ideal I is pseudoprime if and only if $\sqrt{I}$ is prime [4, 4.1]. In [11], Subramanian asks whether this characterization of pseudoprime l -ideals holds in semiprime f -rings. The answer in general is no (even in archimedean f -rings), as witnessed by Example 2.7. However, our next goal is to show that the characterization of pseudoprime l -ideals as those l -ideals I for which $\sqrt{I}$ is prime also holds in a class of normal f -rings.

First, we will give two characterizations of normal f -rings, one of which is the f -ring analogue to a characterization given by Huijsmans in [7]. It should also be noted that the f -ring $C(X)$ is normal if and only if the topological space X is an F -space and that the characterizations of normal f -rings given next are similar to two characterizations of F -spaces given in [3, 14.25]. If P is any prime ideal, the P component of 0 is $O_P = \{a \in A : \exists b \notin P \text{ such that } ab = 0\}$.

THEOREM 2.4. *Let A be a commutative semiprime f -ring with identity element. The following are equivalent.*

- (1) A is normal.
- (2) Every ideal O_P , where P is a (proper) prime l -ideal, is prime.
- (3) Every ideal O_M , where M is a maximal l -ideal, is prime.

PROOF. (1) $\Rightarrow$ (2). Since O_P is a z -ideal, 1.3 implies that it will suffice to show that for all $a \in A$, either $a^+ \in O_P$ or $a^- \in O_P$. Suppose $a \in A$. If $a^+ \notin O_P$ and $a^- \notin O_P$, then $\{a^+\}^d \subseteq P$ and $\{a^-\}^d \subseteq P$. But this would imply $A = \{a^+\}^d + \{a^-\}^d \subseteq P$, contrary to hypothesis.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose A is not normal. Then there is an element $a \in A$ such that $\{a^+\}^d + \{a^-\}^d \neq A$. Let M be a maximal l -ideal containing the l -ideal $\{a^+\}^d + \{a^-\}^d$. Then O_M is a z -ideal such that $a^+ \notin O_M$ and $a^- \notin O_M$. By 1.3, O_M is not prime, contrary to our hypothesis. $\square$

This characterization allows us to make the following observation.

LEMMA 2.5. *If A is a commutative, semiprime normal f -ring with identity element then every minimal prime l -ideal of A is square dominated.*

PROOF. Let P be a minimal prime l -ideal of A . Since the l -ideals containing P form a chain, there is a unique maximal l -ideal M containing P . Now $O_M \subseteq P$ and, by the previous theorem O_M is prime. Therefore, $O_M = P$. We will show O_M

is square dominated. Let $a \in O_M^+$. Then there exists a $b > 0$ such that $b \notin M$ and $ab = 0$. Then $a \wedge b = 0$ and $\{a\}^d + \{b\}^d = A$. Thus, there exists $x \in \{a\}^d, y \in \{b\}^d$ such that $x + y = a \vee 1$. It follows from $y \in \{b\}^d$ and $b \notin M$, that $y \in O_M$. We have $y \in O_M$, and $a \leq y^2$. $\square$

In [9, 3.6] it is shown that if I is an l -ideal of a commutative semiprime f -ring with identity element such that $I = I : \sqrt{I}$, then I is an intersection of primary l -ideals. We will use this fact in the proof of the next theorem.

THEOREM 2.6. *Let A be a commutative semiprime normal f -ring with identity element. The following are equivalent for an l -ideal I .*

- (1) I is pseudoprime.
- (2) The prime l -ideals containing I form a chain.
- (3) $\sqrt{I}$ is prime.

PROOF. We need only show (3) $\Rightarrow$ (1). Suppose that $\sqrt{I}$ is prime. Let P be a minimal prime l -ideal contained in $\sqrt{I}$ and $J = P \cap I$. We will show that J is pseudoprime. Knowing that $\sqrt{J}$ is square dominated (Lemma 2.5), it is not hard to show that $J : \sqrt{J} = (J : \sqrt{J}) : \sqrt{J} : \sqrt{J}$. Let M be the maximal l -ideal containing P . For any $x \in M \setminus J : \sqrt{J}$, there is a primary l -ideal Q_x containing $J : \sqrt{J}$, but not x by the result mentioned above [9, 3.6]. The l -ideals containing P form a chain, so $P = \sqrt{J} \subseteq \sqrt{J} : \sqrt{J} \subseteq \sqrt{Q_x} \subseteq M$. Knowing $Q_x \subseteq M$, it is easy to show that $O_M \subseteq Q_x$ for all x . Now $J : \sqrt{J} = M \cap (\bigcap Q_x)$, so $O_M \subseteq J : \sqrt{J}$. By Theorem 2.4, O_M is a prime l -ideal. Thus $J : \sqrt{J}$ is a pseudoprime l -ideal. Also, since the l -ideals containing O_M form a chain, we have $J : \sqrt{J} \subseteq \sqrt{J}$ or $\sqrt{J} \subseteq J : \sqrt{J}$. In either case, Theorem 2.1 now implies that J is pseudoprime. $\square$

In particular, this result implies that in an f -ring satisfying the hypotheses of the theorem, every primary l -ideal is pseudoprime.

The next example shows that the hypothesis of normality cannot be left out of this theorem. It also shows that the characterization of pseudoprime l -ideals as being those I for which $\sqrt{I}$ is prime does not hold in archimedean f -algebras. In fact, primary l -ideals in archimedean f -algebras are not necessarily pseudoprime.

EXAMPLE 2.7. Let $B = \{f \in C[0, 1] : \exists x_f \in (0, 1) \text{ such that } f(x) = \sum_{i=1}^n a_i x^{r_i}, \text{ where } a_i \in \mathbf{R}, r_i \in \mathbf{Q} \text{ for all } x \in [0, x_f]\}$. Let $P = \{f \in B : f(0) = 0\}$. Then P is a prime l -ideal of B . Let $A = \{(f, g) \in B \times B : f - g \in P\}$. Then as shown in [6], A is a semiprime archimedean f -algebra with identity element. It is not hard to show that every prime l -ideal of A is square dominated.

Let $I = \{(f, g) \in A : f(x), g(x) \leq nx^2 \text{ for some } n, \forall x \in [0, x_f \wedge x_g]\}$. Then I is an l -ideal of A . We will show that I is primary. Suppose $(f, g)(h, k) \in I$, and $(f, g) \notin I$. For every n , either $f(x) \not\leq nx^2$ or $g(x) \not\leq nx^2$ on $[0, a]$ for any $a \in (0, 1)$. Suppose $f(x) \not\leq nx^2$. Then $h(0) = 0$ and $h \in P$. But $h - k \in P$, so $k \in P$. This implies there must exist some N such that $(h, k)^N \in I$. Hence I is primary, and $\sqrt{I}$ is prime. Yet I is not pseudoprime since $(x, 0)(0, x) = (0, 0)$ while $(x, 0) \notin I$ and $(0, x) \notin I$.

To see directly that A is not normal, consider the element $(x, -x) : \{(x, -x)^+\}^d + \{(x, -x)^-\}^d = \{(x, 0)\}^d + \{(0, x)\}^d = \{(f, g) : f, g \in P\} \neq A$. $\square$

We have found some generalizations of the condition that $\sqrt{I}$ be prime that characterize pseudoprime l -ideals in archimedean f -algebras in which minimal prime l -ideals are square dominated. However, each of these conditions are difficult to verify for most l -ideals. If we strengthen our hypotheses to insist that all prime l -ideals of the f -algebra are square dominated, we can give the following generalization to the condition which is a characterization of pseudoprime l -ideals.

THEOREM 2.8. *Let A be an archimedean f -algebra with identity element in which prime l -ideals are square dominated. If I is an l -ideal then I is pseudoprime if and only if when $\{f_{1j}\}_{j=1}^\infty, \dots, \{f_{nj}\}_{j=1}^\infty$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all j , there is a sequence $\{f_{ij}\}_{j=1}^\infty$, for which there exists a positive integer N such that $f_{ij}^N \in I$ for all j .*

PROOF. Let A^* denote the uniform completion of A , and $e: A \rightarrow A^*$ be the embedding.

$\Leftarrow$ Suppose that whenever $\{f_{1j}\}_{j=1}^\infty, \dots, \{f_{nj}\}_{j=1}^\infty$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all j , there is a sequence $\{f_{ij}\}_{j=1}^\infty$, for which there exists a positive integer N such that $f_{ij}^N \in I$ for all j . In A^* , let $M = \{x_1 \cdots x_m: \text{for each } x_i, \forall j \exists a_{ij} \in A \text{ such that } 0 \leq e(a_{ij}) \leq x_i^j \text{ and } a_{ij}^j \notin I\}$. Then M is an m -system. First we will show that $M \cap \{0\} = \emptyset$. Suppose that for $i = 1, 2, \dots, m$, $x_i \in A^*$, such that for every $j \in \mathbb{N}$, there is an $a_{ij} \in A$ such that $0 \leq e(a_{ij}) \leq x_i^j$ and $a_{ij}^j \notin I$. Define

$$f_{ij} = \sum_{k=1}^j \frac{1}{2^k} (a_{ik} \wedge 1) \quad \text{for } i = 1, 2, \dots, m, \quad j = 1, 2, \dots$$

Then $0 \leq (1/2^j)e(a_{ij} \wedge 1) \leq e(f_{ij}) \leq x_i$, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots$, and $\{f_{ij}\}_{j=1}^\infty$ is an increasing positive Cauchy sequence for $i = 1, 2, \dots, m$. Now for each sequence $\{f_{ij}\}_{j=1}^\infty$, there cannot be a natural number N such that $f_{ij}^N \in I$ for all j , because the existence of such an N would imply that $(a_{ij} \wedge 1)^N$, and hence a_{ij}^N , is in I for all j . So by hypothesis, $f_{1j}f_{2j} \cdots f_{mj} \neq 0$ for some j . This, and the fact that $0 \leq e(f_{1j})e(f_{2j}) \cdots e(f_{mj}) \leq x_1x_2 \cdots x_m$ implies $x_1x_2 \cdots x_m \neq 0$. Therefore, $M \cap \{0\} = \emptyset$.

Thus there is a prime l -ideal in A^* containing $\{0\}$ and disjoint from M . Let P be a minimal prime l -ideal with this property. We will show $P \cap e(A) \subseteq e(I)$. Let $f \in A^+$ such that $e(f) \in P \cap e(A)$.

By 1.2, there is a $g \notin P$ such that $e(f)g = 0$. In A , $\{g\}_A^d = \{x \in A: xg = 0\}$ is a semiprime l -ideal and Lemma 2.1(1) implies it is square dominated. So $f \leq f_1^2$ for some $f_1 \in A^+$ with $f_1^2 \in \{g\}_A^d$. Since $\{g\}_A^d$ is semiprime, $f_1 \in \{g\}_A^d$. As before, $f_1 \leq f_2^2$ for some $f_2 \in A^+$ with $f_2^2 \in \{g\}_A^d$. Again $f_2 \in \{g\}_A^d$. So $f \leq f_1^2 \leq f_2^4$. Continuing this for $i = 3, 4, \dots$, we find $f \leq f_1^2 \leq f_2^4 \leq \cdots \leq f_i^{2^i} \leq \cdots$. Define

$$h_j = \sum_{i=1}^j \frac{1}{2^i} (f_i \wedge 1) \quad \text{for } j = 1, 2, \dots$$

Then $\{h_j\}_{j=1}^\infty$ is an increasing Cauchy sequence in A and $\{e(h_j)\}_{j=1}^\infty$ converges to an element $h \in A^*$. Now $hg = 0$ and $g \notin P$, implying $h \in P$.

We assert that there exists some positive integer M such that

$$(1/2^M)(f_M \wedge 1)^{2^M} \in I.$$

For if not, $0 \leq (1/2^i)(f_i \wedge 1)^{2^i} \leq (f_i \wedge 1)^{i^2}$ would imply $(f_i \wedge 1)^{i^2} = ((f_i \wedge 1)^i)^i \notin I$ for all $i \neq 3$. We would then have

$$0 \leq (1/2^{i^2})(f_i \wedge 1)^i \leq h^i \quad \text{and} \quad ((1/2^{i^2})(f_i \wedge 1)^i)^i \notin I \quad \text{for all } i.$$

But this would imply $h \in M$, contrary to the fact that $M \cap P = \emptyset$. So there exists some positive integer M such that $(1/2^M)(f_M \wedge 1)^{2^M} \in I$.

We now know $(1/2^M)(f \wedge 1) \leq (1/2^M)(f_M \wedge 1)^{2^M}$ implies $(1/2^M)(f \wedge 1) \in I$. Therefore $(f \wedge 1) \in I$ and $f = (f \wedge 1)(f \vee 1) \in I$. We have $P \cap e(A) \subseteq e(I)$ and $P \cap e(A)$ is prime in $e(A)$. Therefore $e(I)$ is pseudoprime in A^* . This implies I is pseudoprime in A .

$\Rightarrow$ Suppose that I is pseudoprime and that $\{f_{1j}\}_{j=1}^\infty, \dots, \{f_{nj}\}_{j=1}^\infty$ are increasing positive Cauchy sequences in A with $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all j . Each of the sequences $\{e(f_{ij})\}_{j=1}^\infty$ converges to some element f_i in A^* . Let P be a prime l -ideal contained in I . Then $M = \{e(a) : a \in A, a \notin P\}$ is an m -system in A^* . So there is a prime l -ideal P^* of A^* such that $P \subseteq P^*$ and $P^* \cap M = \emptyset$. Now $e(f_1)e(f_2) \cdots e(f_n) = 0$ and so $e(f_m) \in P^*$ for some m . Thus $e(f_{mj}) \leq e(f_m)$ implies that $e(f_{mj}) \in P^* \cap e(A) = e(P)$ for all j . $\square$

It is not difficult to show directly that in a uniformly complete f -ring, the property characterizing a pseudoprime l -ideal I given in this theorem is equivalent to the property that $\sqrt{I}$ is prime.

Finally we show that the hypothesis that minimal prime l -ideals be square dominated cannot be dropped or generalized in any way in any of our theorems characterizing pseudoprime l -ideals.

LEMMA 2.9. *Let A be a semiprime f -ring. If in A , a pseudoprime l -ideal I is characterized by being an l -ideal that satisfies any one of the following conditions, then every minimal prime l -ideal of A is square dominated.*

- (1) $\bigcap_{n=1}^\infty \langle I^n \rangle$ is prime.
- (2) $\langle I\sqrt{I} \rangle$ is pseudoprime.
- (3) $I : \sqrt{I}$ is pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is prime.
- (4) The prime l -ideals containing I form a chain.
- (5) $\sqrt{I}$ is prime.

Also, if A is an archimedean f -ring, and if in A , a pseudoprime l -ideal I is characterized by being an l -ideal that satisfies the following condition, then every minimal prime l -ideal of A is square dominated.

- (6) Whenever $\{f_{1j}\}_{j=1}^\infty, \dots, \{f_{nj}\}_{j=1}^\infty$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all j , there is a sequence $\{f_{ij}\}_{j=1}^\infty$, for which there exists a positive integer N such that $f_{ij}^N \in I$ for all j .

PROOF. Let P be a minimal prime l -ideal. Note that in each case it will suffice to show that $\langle P^2 \rangle$ is pseudoprime since $\langle P^2 \rangle \subseteq P$ and P being a minimal prime l -ideal will then imply $\langle P^2 \rangle = P$.

If characterization (1) holds, then $\bigcap_{n=1}^\infty \langle P^n \rangle$ is a prime l -ideal contained in $\langle P^2 \rangle$. So $\langle P^2 \rangle$ is pseudoprime.

Characterization (2) implies $\langle P\sqrt{P} \rangle = \langle P^2 \rangle$ is pseudoprime.

Suppose characterization (3) holds. Note that $\langle P^2 \rangle : \sqrt{\langle P^2 \rangle} = \langle P^2 \rangle : P \supseteq \sqrt{\langle P^2 \rangle} = P$. So $\langle P^2 \rangle : \sqrt{\langle P^2 \rangle}$ is pseudoprime. Then characterization (3) implies $\langle P^2 \rangle$ is pseudoprime.

Suppose characterization (4) holds. Every prime l -ideal containing $\langle P^2 \rangle$ also contains P . So the prime l -ideals containing $\langle P^2 \rangle$ form a chain. Characterization (4) implies $\langle P^2 \rangle$ is pseudoprime.

Suppose characterization (5) holds. Then $\sqrt{\langle P^2 \rangle} = P$ is prime, implying that $\langle P^2 \rangle$ is pseudoprime.

Finally, suppose that A is an archimedean f -algebra and that characterization (6) holds. Suppose that $\{f_{1j}\}_{j=1}^\infty, \dots, \{f_{nj}\}_{j=1}^\infty$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all j . Then for some m , $f_{mj} \in P$ for all j . But then $f_{mj}^2 \in \langle P^2 \rangle$ for all j . So characterization (6) implies that $\langle P^2 \rangle$ is pseudoprime. $\square$

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