SOME CONSEQUENCES OF THE STANDARD POLYNOMIAL

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ABSTRACT. The standard polynomial of degree m is the polynomial $\sum\{\text{sign}(\rho)x_{\rho(1)}x_{\rho(2)}\cdots x_{\rho(m)}|\rho\in S_m\}$, where S_m is the symmetric group on m letters. We show that the polynomial

$$\sum\{\text{sign}(\rho\sigma)x_{\rho(1)}y_{\sigma(1)}x_{\rho(2)}y_{\sigma(2)}\cdots x_{\rho(m)}y_{\sigma(m)}|\rho,\sigma\in S_m\}$$

is a consequence of the standard polynomial of degree m . We also show that certain polynomials of the form $\sum\{\text{sign}(\rho)x_{\rho(1)}x_{\rho(2)}\cdots x_{\rho(n)}|\rho\in Q\}$, where $n\geq m$ and Q is a suitable subset of S_n , are consequences of the standard polynomial of degree m .

1. Introduction. Two of the most important polynomials in the theory of polynomial identity rings are the *standard polynomial* and the *Capelli polynomial*, which are defined, respectively, for each positive integer m , by

$$\mathcal{S}_m(x_1, \dots, x_m) = \sum_{\rho \in S_m} (\text{sign } \rho) x_{\rho(1)} x_{\rho(2)} \cdots x_{\rho(m)},$$

$$\mathcal{C}_m(x_1, \dots, x_m, y_1, \dots, y_m) = \sum_{\rho \in S_m} (\text{sign } \rho) x_{\rho(1)} y_1 x_{\rho(2)} y_2 \cdots x_{\rho(m)} y_m.$$

The Amitsur-Levitzki Theorem says that $\mathcal{S}_{2n}$ is a polynomial identity for $M_n(K)$, the ring of $n \times n$ matrices over a field K , but $\mathcal{S}_{2n-1}$ is not. Razmyslov has shown that $\mathcal{C}_{n^2+1}$ is a polynomial identity for $M_n(K)$, but $\mathcal{C}_{n^2}$ is not.

This paper was motivated by a question of Formanek. Define the *double Capelli polynomial* by

$$\mathcal{D}_m(x_1, \dots, x_m, y_1, \dots, y_m) = \sum_{\rho, \sigma \in S_m} (\text{sign } \rho\sigma) x_{\rho(1)} y_{\sigma(1)} \cdots x_{\rho(m)} y_{\sigma(m)}.$$

Formanek pointed out that $\mathcal{D}_{2n-1}$ is not a polynomial identity for $M_n(K)$ and asked for the least integer m such that $\mathcal{D}_m$ is a polynomial identity for $M_n(K)$. We will show (Theorem 2) that $m = 2n$ is the least such integer. This result will be a corollary of our main result (Theorem 1), which says that certain polynomials, one of which is $\mathcal{D}_m$, are consequences of the standard polynomial $\mathcal{S}_m$. (A polynomial $\mathcal{F}$ is a *consequence* of a polynomial $\mathcal{G}$ if it lies in the T -ideal generated by $\mathcal{G}$.)

These polynomials are defined as follows. Let $x_1, \dots, x_m, y_1, \dots, y_t$ be noncommuting variables over K , and let $w_1, \dots, w_u$ be monomials in $y_1, \dots, y_t$ such that $w_1 \dots w_u$ is a reordering of $y_1 \dots y_t$. Define a polynomial

$$\mathcal{F} = \mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_u)$$

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by

$$\mathcal{F} = \sum (\text{sign } \mu) x_{\rho(1)} \cdots x_{\rho(n_1)} w_{\sigma(1)} x_{\rho(n_1+1)} \cdots x_{\rho(n_1+n_2)} w_{\sigma(2)} x_{\rho(n_1+n_2+1)} \cdots x_{\rho(n_1+\cdots+n_u)} w_{\sigma(u)} x_{\rho(n_1+\cdots+n_u+1)} \cdots x_{\rho(n_1+\cdots+n_{u+1})},$$

where the sum is over all $\rho \in S_m$, $\sigma \in S_u$, and all choices of integers $n_1, \dots, n_u \geq 1$, $n_{u+1} \geq 0$, such that $n_1 + \cdots + n_{u+1} = m$, and the sign attached to each such monomial μ is ± 1 according to whether μ is an even or an odd permutation of the “underlying” variables $x_1, \dots, x_m, y_1, \dots, y_t$. Theorem 1 states that $\mathcal{F}$ is a consequence of the standard polynomial $\mathcal{S}_m$ of degree m .

A few comments about the polynomial $\mathcal{F}$ are in order. First, it is a homogeneous multilinear polynomial in the variables $x_1, \dots, x_m, y_1, \dots, y_t$, and might more properly be written $\mathcal{F}_\pi(x_1, \dots, x_m, y_1, \dots, y_t)$, where $\pi = (w_1, \dots, w_u)$ is a partition of a reordering of the product $y_1 \cdots y_t$ into u subwords. Second, $\mathcal{F}$ is a summand of the standard polynomial $\mathcal{S}_{m+t}$, in the sense that

$$\mathcal{F} = \sum_{\rho \in Q} (\text{sign } \rho) z_{\rho(1)} \cdots z_{\rho(m+t)},$$

where $\{x_1, \dots, x_m, y_1, \dots, y_t\} = \{z_1, \dots, z_{m+t}\}$ and Q is a certain subset of S_{m+t} . Third, the sign attached to a monomial μ depends only on the parity of the lengths of $w_1, \dots, w_u$. Fourth, the condition on $n_1, \dots, n_{u+1}$ means that the sum is precisely over those monomials in $x_1, \dots, x_m, w_1, \dots, w_u$ having the property that each w_i is immediately preceded by some x_j . This point is important in the proof of the main theorem.

2. Proofs of the theorems.

THEOREM 1. $\mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_u)$ is a consequence of the standard polynomial $\mathcal{S}_m$ of degree m .

PROOF. The proof is by induction on the pair (t, u) associated with $\mathcal{F}$, where $(t_1, u_1) < (t, u)$ if $t_1 < t$, or $t_1 = t$ and $u_1 < u$.

The induction begins with $(1, 1)$, when $w_1 = y_1$. Then

$$(1) \quad \mathcal{S}_{m+1}(x_1, \dots, x_m, y_1) = (-1)^m y_1 \mathcal{S}_m(x_1, \dots, x_m) + \mathcal{F}(x_1, \dots, x_m, y_1 | w_1),$$

since $\mathcal{F}(x_1, \dots, x_m, y_1 | w_1)$ is the sum, with the right sign, of all the monomials in $\mathcal{S}_{m+1}(x_1, \dots, x_m, y_1)$ in which y_1 is immediately preceded by some x_i . Since $\mathcal{S}_{m+1}$ is a consequence of $\mathcal{S}_m$, (1) shows that $\mathcal{F}(x_1, \dots, x_m, y_1 | w_1)$ is also a consequence of $\mathcal{S}_m$.

For the inductive step, assume that $\mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_u)$ is a consequence of $\mathcal{S}_m$ whenever $(t_1, u_1) < (t, u)$. There are three cases, which are not mutually exclusive.

Case 1. Some monomial w_i has length ≥ 3 in terms of $y_1, \dots, y_t$. To avoid cumbersome notation, we assume that the monomial in question is $w_u = * \cdots * y_{t-2} y_{t-1} y_t$. Now

$$(2) \quad \begin{aligned} \mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_{t-3}, y_{t-2}, y_{t-1}, y_t | w_1, \dots, w_u) \\ = \mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_{t-3}, (y_{t-2} y_{t-1} y_t) | w_1, \dots, w_u), \end{aligned}$$

since both functions are sums of exactly the same monomials, with the same signs. However, the pair (t, u) is associated with the first polynomial, while the pair

$(t-2, u)$ is associated with the second polynomial. Thus the second polynomial in (2) is a consequence of $\mathcal{S}_m$ by the inductive hypothesis, and so the first polynomial in (2) is also a consequence of $\mathcal{S}_m$.

Case 2. Some monomial w_i has length exactly 2. Again, for notational simplicity, we assume that the monomial is $w_u = y_{t-1}y_t$. Consider the equation

$$(3) \quad \begin{aligned} & \mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_u) \\ &= \sum_{1 \leq i \leq m} \mathcal{F}(x_1, \dots, x_{i-1}, x_i w_u, x_{i+1}, \dots, x_m, y_1, \dots, y_{t-2} | w_1, \dots, w_{u-1}) \\ & \quad - \sum_{1 \leq j \leq u} \mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_{j-1}, w_u w_j, w_{j+1}, \dots, w_{u-1}). \end{aligned}$$

To see that the above equation is valid, note that when the first function is written as an alternating sum of monomials, w_u is immediately preceded in each monomial by some x_i and thus occurs exactly once, with the proper sign, in the first summation of the right-hand side of the equation. Conversely, every monomial in this first summation occurs with the proper sign in the first function, *except* those monomials where $x_i w_u$ immediately precedes some w_j . Subtracting the final summation eliminates these monomials and verifies equation (3).

The $\mathcal{F}$'s in the first summation have associated pair $(t-2, u-1)$ and the $\mathcal{F}$'s in the second summation have associated pair $(t, u-1)$. Hence both are consequences of $\mathcal{S}_m$, by the inductive hypothesis, and so $\mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | w_1, \dots, w_u)$ is also a consequence of $\mathcal{S}_m$.

Case 3. All the w_i 's have length 1. Then, up to a change of sign, we may suppose that $w_1 = y_1, \dots, w_t = y_t$. Consider $\mathcal{S}_{m+t}(x_1, \dots, x_m, y_1, \dots, y_t)$, the standard polynomial. When it is expanded, every monomial in the expansion has the form

$$\mu_0(y)\nu_1(x)\mu_1(y)\nu_2(x)\mu_2(y)\cdots\nu_r(x)\mu_r(y)\nu_{r+1}(x),$$

where $r \geq 0$, $\mu_0(y), \dots, \mu_r(y)$ are monomials in $y_1, \dots, y_t$, $\nu_1(x), \dots, \nu_{r+1}(x)$ are monomials in $x_1, \dots, x_m$, and all these monomials are nontrivial except possibly $\mu_0(y)$ and $\nu_{r+1}(x)$. The monomials with the same $\mu_0(y)$ and unordered set $\{\mu_1(y), \dots, \mu_r(y)\}$ together make a contribution to $\mathcal{S}_{m+t}$ equal to

$$\pm \mu_0(y) \mathcal{F}(x_1, \dots, x_m, y_{i_1}, \dots, y_{i_s} | \mu_1(y), \dots, \mu_r(y)).$$

One such summand of $\mathcal{S}_{m+t}$ is $\mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | y_1, \dots, y_t)$, which has associated pair (t, t) , but all of the remaining $\mathcal{F}$'s in the summation have associated pair $(s, r) < (t, t)$, and thus are consequences of $\mathcal{S}_m$, by the inductive hypothesis. Thus $\mathcal{F}(x_1, \dots, x_m, y_1, \dots, y_t | y_1, \dots, y_t)$ is also a consequence of $\mathcal{S}_m$, completing the induction. $\square$

The answer to Formanek's question is a corollary to Theorem 1.

THEOREM 2. *Let*

$$\mathcal{D}_m(x_1, \dots, x_m, y_1, \dots, y_m) = \sum_{\rho, \sigma \in S_m} (\text{sign } \rho \sigma) x_{\rho(1)} y_{\sigma(1)} \cdots x_{\rho(m)} y_{\sigma(m)}.$$

Then $\mathcal{D}_{2n}$ is a polynomial identity for $M_n(K)$, the ring of $n \times n$ matrices over a field K , but $\mathcal{D}_{2n-1}$ is not.

PROOF. $\mathcal{D}_{2n}(x_1, \dots, x_{2n}, y_1, \dots, y_{2n}) = \mathcal{F}(x_1, \dots, x_{2n}, y_1, \dots, y_{2n} | y_1, \dots, y_{2n})$. Since $\mathcal{F}$ is a consequence of $\mathcal{S}_{2n}$ and $\mathcal{S}_{2n}$ is a polynomial identity for $M_n(K)$ (Amitsur-Levitzki Theorem), it follows that $\mathcal{D}_{2n}$ is a polynomial identity for $M_n(K)$.

To see that $\mathcal{D}_{2n-1}$ is not a polynomial identity for $M_n(K)$, consider the substitution

$$\begin{aligned}(u_1, \dots, u_{2n-1}) &= (e_{11}, e_{22}, \dots, e_{nn}, e_{nn-1}, e_{n-1n-2}, \dots, e_{21}) \\ (v_1, \dots, v_{2n-1}) &= (e_{12}, e_{23}, \dots, e_{n-1n}, e_{nn}, e_{n-1n-1}, \dots, e_{11}),\end{aligned}$$

where the e_{ij} are the standard matrix units. The only nonzero monomials in $\mathcal{D}_{2n-1}(u_i, v_i)$ are the $2n-1$ even cyclic permutations of $u_1 v_1 \dots u_{2n-1} v_{2n-1}$, and they all have sign $+1$. Thus

$$\begin{aligned}\mathcal{D}_{2n-1}(u_i, v_i) &= e_{11} + e_{22} + \dots + e_{nn} + e_{nn} + \dots + e_{22} \\ &= 2I - e_{11}. \quad \square\end{aligned}$$

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ADDED IN PROOF: Recently I was told that Professor A. Giambruno and Professor S. K. Sehgal proved the result of Theorem 2.

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