CONSTRUCTIVE REFLEXIVITY OF A UNIFORMLY CONVEX BANACH SPACE

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ABSTRACT. In this paper we consider a question about reflexivity of a Banach space within the framework of Bishop's constructive mathematics and we give a partially affirmative answer to the question set by Bishop: "Is every uniformly convex Banach space reflexive?"

In this paper we consider a question about reflexivity of a Banach space. Classically, the question has an answer that every uniformly convex Banach space is reflexive; see [6, Theorem 8.2.1; 7, Theorem V.2.2]. However, within the framework of Bishop's constructive mathematics, the question is more sensitive, because, for a bounded linear functional f on a normed linear space E , the norm

$$\|f\| \equiv \sup\{|f(x)| : x \in E, \|x\| \leq 1\}$$

does not always exist and hence we cannot treat the dual space E^* of E as classically. In particular, we do not know that the set $E' \subset E^*$ of normable linear functionals (i.e. linear functionals whose norms exist) on E is a linear space. Therefore Bishop called a Banach space E *reflexive* if

(A) E' is closed with respect to addition, that is, E' is a linear space;

(B) every normable linear functional on E' is of the form $f \mapsto f(x_0)$ for some x_0 in E , and set a question: "Is every uniformly convex Banach space reflexive?"; see [1, p. 295; 2, p. 393] and, for the foundations of constructive mathematics, see [1, 2, 3, 4]. The main result (Theorem 2) of the present paper gives the question a partially affirmative answer: *every uniformly convex complete normed linear space which satisfies the condition (A) is reflexive, that is, the condition (B) holds.*

Let $\mathbf{F}$ denote either the real or the complex number field and let E be a normed linear space over $\mathbf{F}$. Then E is said to be *uniformly convex* if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, called the *modulus of convexity* of E , such that for each $x, y \in E$ with $\|x\| \leq 1$ and $\|y\| \leq 1$,

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon) \text{ whenever } \|x-y\| \geq \varepsilon.$$

Let f be a bounded linear functional on E and let V be a linear subset of E . Then the norm of f as a linear functional on V is

$$\|f\|_V \equiv \sup\{|f(x)| : x \in V, \|x\| \leq 1\},$$

when it exists.

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We begin with the following lemma which plays a crucial role in the proof of theorem.

LEMMA 1. *Let E be a normed linear space over $\mathbf{F}$ which satisfies the condition (A) and let Y be a finite-dimensional subspace of E' . Then for each $\varepsilon > 0$ there exists a finite-dimensional subspace V of E such that*

$$\|f\|_V \geq (1 - \varepsilon)\|f\| \quad (f \in Y).$$

PROOF. Let $S_Y \equiv \{f \in Y: \|f\| \leq 1\}$ and let η be a positive number with $\eta(5 + \eta) < \varepsilon$. Then since S_Y is a compact subset of E' there exists an η approximation $\{g_1, \dots, g_m\}$ to S_Y . Also since each g_i is normable, there exists u_i in E such that $\|u_i\| = 1$ and $|g_i(u_i)| > \|g_i\| - \eta$. By Lemma 2.5 of [2, p. 308], there exists a finite-dimensional subspace V of E such that $d(u_i, V) < \eta$ for $i = 1, \dots, m$. Let f be a linear functional in Y with $\|f\| \leq 1$. Then there exists i and $v_i \in V$ such that $\|f - g_i\| < \eta$ and $\|u_i - v_i\| < \eta$. Hence we have

$$\begin{aligned} \|f\| &\leq \|g_i\| + \|f - g_i\| < |g_i(u_i)| + \eta + \eta \\ &\leq |f(v_i)| + |g_i(v_i) - f(v_i)| + |g_i(u_i) - g_i(v_i)| + 2\eta \\ &\leq \|f\|_V \|v_i\| + \|g_i - f\| \|v_i\| + \|g_i\| \|u_i - v_i\| + 2\eta \\ &< \|f\|_V (1 + \eta) + \eta(1 + \eta) + \eta + 2\eta = \|f\|_V + \eta\|f\|_V + 4\eta + \eta^2 \\ &\leq \|f\|_V + \eta\|f\| + 4\eta + \eta^2 \leq \|f\|_V + \eta(5 + \eta) < \|f\|_V + \varepsilon. \end{aligned}$$

For arbitrary $f \in Y$ and each positive number t , putting $f_t \equiv f / \max\{t, \|f\|\}$, we have $\|f_t\| < \|f_t\|_V + \varepsilon$ and hence $\|f\| < \|f\|_V + \varepsilon \max\{t, \|f\|\}$. Therefore, letting $t \rightarrow 0$, we obtain

$$\|f\| \leq \|f\|_V + \varepsilon\|f\|. \quad \square$$

Next, we prove a constructive substitute for Helly's Theorem; see [6, Theorem 4.10.1; 7, Theorem IV.6.5].

THEOREM 1. *Let E be a normed linear space over $\mathbf{F}$ which satisfies the condition (A) and let M be a positive number. Suppose that $f_1, \dots, f_n$ are in E' and $c_1, \dots, c_n$ are in $\mathbf{F}$. Then the following are equivalent:*

- (1) *For each $\varepsilon > 0$ there exists x_0 in E such that $\|x_0\| \leq M + \varepsilon$ and $|f_k(x_0) - c_k| < \varepsilon$ for $k = 1, \dots, n$;*
- (2) *$|\sum_{k=1}^n \lambda_k c_k| \leq M \|\sum_{k=1}^n \lambda_k f_k\|$ for every $\lambda_1, \dots, \lambda_n$ in $\mathbf{F}$.*

PROOF. If (1) holds, then for each $\varepsilon > 0$ there exists x_0 in E such that $\|x_0\| \leq M + \varepsilon$ and $|f_k(x_0) - c_k| < \varepsilon$ for $k = 1, \dots, n$. It follows that for any $\lambda_1, \dots, \lambda_n$ in $\mathbf{F}$ we have

$$\begin{aligned} \left| \sum_{k=1}^n \lambda_k c_k \right| &\leq \left| \sum_{k=1}^n \lambda_k f_k(x_0) \right| + \left| \sum_{k=1}^n \lambda_k (c_k - f_k(x_0)) \right| \\ &\leq \left\| \sum_{k=1}^n \lambda_k f_k \right\| \|x_0\| + \varepsilon \sum_{k=1}^n |\lambda_k| \\ &\leq (M + \varepsilon) \left\| \sum_{k=1}^n \lambda_k f_k \right\| + \varepsilon \sum_{k=1}^n |\lambda_k|. \end{aligned}$$

Since ε is arbitrary, we conclude that (1) implies (2). Conversely, suppose that (2) holds and let Y be a finite-dimensional subspace of E' such that $d(f_k, Y) < \varepsilon/(3n(M+\varepsilon))$ for $k = 1, \dots, n$. Then, by Lemma 1, there exists a finite-dimensional subspace V of E such that

$$\|g\|_V \geq \frac{M}{M+\varepsilon} \|g\| \quad (g \in Y).$$

Let

$$K \equiv \{x \in V : \|x\| \leq M + \varepsilon\}$$

and, for each $k = 1, \dots, n$, let g_k be a functional in Y with

$$\|f_k - g_k\| < \frac{\varepsilon}{3n(M+\varepsilon)}.$$

Then, since K is compact and convex,

$$D \equiv \{(g_1(x), \dots, g_n(x)) : x \in K\}$$

is a totally bounded convex subset of $\mathbf{F}^n$ with the norm

$$\|(\lambda_1, \dots, \lambda_n)\| \equiv (|\lambda_1|^2 + \dots + |\lambda_n|^2)^{1/2}.$$

Suppose that $d \equiv d(c, D) > \varepsilon/3$ where $c \equiv (c_1, \dots, c_n)$. Then, by the separation theorem [2, Theorem 4.3 of p. 336 or 5, Theorem 2], there exists $(\lambda_1, \dots, \lambda_n)$ in $\mathbf{F}^n$ such that

$$|\lambda_1|^2 + \dots + |\lambda_n|^2 = 1$$

and

$$\operatorname{Re} \sum_{k=1}^n \lambda_k c_k \geq \operatorname{Re} \sum_{k=1}^n \lambda_k g_k(x) + d \quad (x \in K).$$

Since K is balanced, we have

$$\begin{aligned} \operatorname{Re} \sum_{k=1}^n \lambda_k c_k &\geq \sup_{x \in K} \left| \operatorname{Re} \sum_{k=1}^n \lambda_k g_k(x) \right| + d \\ &= (M + \varepsilon) \left\| \operatorname{Re} \sum_{k=1}^n \lambda_k g_k \right\|_V + d = (M + \varepsilon) \left\| \sum_{k=1}^n \lambda_k g_k \right\|_V + d \\ &\geq M \left\| \sum_{k=1}^n \lambda_k g_k \right\| + d \quad (\text{by our choice of } V) \\ &\geq M \left(\left\| \sum_{k=1}^n \lambda_k f_k \right\| - \left\| \sum_{k=1}^n \lambda_k (g_k - f_k) \right\| \right) + d \\ &\geq M \left(\left\| \sum_{k=1}^n \lambda_k f_k \right\| - \sum_{k=1}^n |\lambda_k| \|g_k - f_k\| \right) + d \\ &> M \left(\left\| \sum_{k=1}^n \lambda_k f_k \right\| - n \frac{\varepsilon}{3n(M+\varepsilon)} \right) + d \\ &> M \left\| \sum_{k=1}^n \lambda_k f_k \right\| - \frac{\varepsilon}{3} + d > \left\| \sum_{k=1}^n \lambda_k c_k \right\|. \end{aligned}$$

This contradiction implies $d(c, D) \leq \varepsilon/3$. Hence there exists x_0 in K such that $|g_k(x_0) - c_k| < \varepsilon/2$ for $k = 1, \dots, n$. Therefore we have, for each $k = 1, \dots, n$,

$$\begin{aligned} |f_k(x_0) - c_k| &\leq |f_k(x_0) - g_k(x_0)| + |g_k(x_0) - c_k| \\ &< \|f_k - g_k\| \|x_0\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{3n(M + \varepsilon)}(M + \varepsilon) + \frac{\varepsilon}{2} < \varepsilon. \quad \square \end{aligned}$$

COROLLARY 1. *Let E be a normed linear space over $\mathbf{F}$ which satisfies the condition (A) and let F_0 be a nonzero normable linear functional on E' . Then for any $f_1, \dots, f_n$ in E' and any $\varepsilon > 0$ there exists x_0 in E such that*

- (1) $\|x_0\| \leq \|F_0\| + \varepsilon$,
- (2) $|f_k(x_0) - F_0(f_k)| < \varepsilon$ ($k = 1, \dots, n$).

PROOF. Apply Theorem 1 with $c_k \equiv F_0(f_k)$, $k = 1, \dots, n$, and $M \equiv \|F_0\|$. $\square$
Now, we can prove our main result.

THEOREM 2. *Let E be a uniformly convex complete normed linear space over $\mathbf{F}$ which satisfies the condition (A). Then E is reflexive, that is, the condition (B) holds.*

PROOF. Suppose that F_0 is a nonzero normable linear functional on E' . We may assume without loss of generality that $\|F_0\| = 1$. Let $f_0, f_1, \dots$ be a sequence in E' , where $\|f_k\| = 1$ for all k and $F_0(f_k) > 1 - 1/k$ for $k = 1, 2, \dots$. Then for each positive integer n , applying Corollary 1 with $\varepsilon \equiv 1/n$, we construct x_n in E such that

- (1) $\|x_n\| \leq 1 + 1/n$,
- (2) $|f_k(x_n) - F_0(f_k)| < 1/n$ ($k = 0, \dots, n$).

In particular, for each $N = 0, 1, \dots$, we have

$$|f_N(x_n) - F_0(f_N)| < 1/n \quad (n > N).$$

We show that $\{x_n\}$ is a Cauchy sequence. Let δ be a modulus of convexity of E and, for each $\varepsilon > 0$, choose a positive integer N so that $3/(N + 1) < \delta(\varepsilon/2)$. For all n and m with $m, n \geq N$, suppose that $\|x_m - x_n\| > \varepsilon$. Then we have

$$\begin{aligned} \left\| \frac{x_m}{1 + 1/N} \right\| &\leq \frac{1 + 1/m}{1 + 1/N} \leq 1, \\ \left\| \frac{x_n}{1 + 1/N} \right\| &\leq \frac{1 + 1/n}{1 + 1/N} \leq 1 \end{aligned}$$

and

$$\left\| \frac{x_m - x_n}{1 + 1/N} \right\| > \frac{\varepsilon}{1 + 1/N} \geq \frac{\varepsilon}{2}.$$

Hence we obtain

$$\left\| \frac{x_m + x_n}{2(1 + 1/N)} \right\| \leq 1 - \delta\left(\frac{\varepsilon}{2}\right)$$

or

$$\begin{aligned} \|x_m + x_n\| &\leq 2 \left(1 + \frac{1}{N}\right) \left(1 - \delta\left(\frac{\varepsilon}{2}\right)\right) \\ &< 2 \left(1 + \frac{1}{N}\right) \left(1 - \frac{3}{N + 1}\right) \\ &= 2 \left(1 - \frac{2}{N}\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_m + x_n\| &= \|f_N\| \|x_m + x_n\| \\ &\geq |f_N(x_m) + f_N(x_n)| \\ &\geq 2|F_0(f_N)| - |F_0(f_N) - f_N(x_m)| - |F_0(f_N) - f_N(x_n)| \\ &> 2\left(1 - \frac{1}{N}\right) - \frac{1}{m} - \frac{1}{n} \geq 2\left(1 - \frac{1}{N}\right) - \frac{1}{N} - \frac{1}{N} = 2\left(1 - \frac{2}{N}\right).\end{aligned}$$

This contradiction shows that $\|x_m - x_n\| \leq \varepsilon$ for all $m, n \geq N$. Hence $\{x_n\}$ is a Cauchy sequence. Let x_0 be its limit. Then we have

$$1 - \frac{2}{n} < F_0(f_n) + (f_n(x_n) - F_0(f_n)) = f_n(x_n) \leq \|x_n\| \leq 1 + \frac{1}{n}$$

and hence, letting $n \rightarrow \infty$, we obtain $\|x_0\| = \lim_{n \rightarrow \infty} \|x_n\| = 1$. Also since $|f_N(x_n) - f_0(f_N)| < 1/n$ for all $n > N$ ($N = 0, 1, \dots$), we have

$$|f_N(x_0) - F_0(f_N)| = \lim_{n \rightarrow \infty} |f_N(x_n) - F_0(f_N)| = 0 \quad (N = 0, 1, \dots).$$

Let $f \in E'$. Then for each $\varepsilon > 0$, either $\|f\| < \varepsilon/2$ or $0 < \|f\|$; in the former case, we have

$$|f(x_0) - F_0(f)| \leq |f(x_0)| + |F_0(f)| \leq 2\|f\| < \varepsilon.$$

In the latter case, applying the above procedure with $f_0 \equiv f/\|f\|$, we have a vector $y_0 \in E$ such that $\|y_0\| = 1$ and $f_n(y_0) = F_0(f_n)$ for $n = 0, 1, \dots$. Suppose that $\eta \equiv \|x_0 - y_0\| > 0$. Then we have $\|(x_0 + y_0)/2\| < 1 - \delta(\eta)$, so $\|x_0 + y_0\| < 2(1 - \delta(\eta))$. On the other hand, since $f_n(y_0) = F_0(f_n)$ for $n = 0, 1, \dots$, we have

$$\|x_0 + y_0\| \geq |f_n(x_0 + y_0)| = 2F_0(f_n) > 2(1 - 1/n) \quad (n = 1, 2, \dots)$$

and hence $\|x_0 + y_0\| = 2$. This contradiction implies that $y_0 = x_0$. Therefore we obtain

$$|f(x_0) - F_0(f)| = \|f\| |g_1(y_0) - F_0(g_1)| = 0 < \varepsilon.$$

Thus, in either case, since ε is arbitrary, we have $f(x_0) = F_0(f)$. For a nonzero normable linear functional F on E' , we denote by $\kappa(F)$ the vector of E such that $\|\kappa(F)\| = \|F\|$ and $f(\kappa(F)) = F(f)$ for all $f \in E'$ [2, p. 323, (3.23)]. Let F be an arbitrary normable linear functional on E' and let $\{z_n\}$ be a sequence of vectors in E such that

$$z_n \equiv 0 \quad \text{if } \|F\| < 1/k \text{ for all } k = 1, \dots, n$$

and

$$z_n \equiv \kappa(F) \quad \text{if } \|F\| > 0 \text{ for some } k = 1, \dots, n.$$

Then $\{z_n\}$ is a Cauchy sequence which satisfies $|\|z_n\| - \|F\|| < 1/n$ and $|f(z_n) - F(f)| \leq \|f\|/n$ for all f in E' . Let z_0 be the limit of $\{z_n\}$. Then $\|z_0\| = \|F\|$ and $f(z_0) = F(f)$ for all $f \in E'$. $\square$

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