THE ZALCMAN CONJECTURE FOR CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. Let S be the class of functions $f(z) = z + \cdots$ analytic and univalent in the unit disk D . For $f(z) = z + a_2z^2 + \cdots \in S$, Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ ($n = 2, 3, \dots$). This conjecture is verified for $n \geq 4$ and close-to-convex functions.

Let S be the class of functions $f(z)$ analytic and one-to-one in the unit disk $D = \{z: |z| < 1\}$ and normalization $f(0) = f'(0) - 1 = 0$. Recently, Zalcman conjectured that

$$(1) \quad |a_n^2 - a_{2n-1}| \leq (n-1)^2$$

for each $n \geq 2$ and each $f(z) = z + a_2z^2 + \cdots \in S$, which implies the famous Bieberbach conjecture $|a_n| \leq n$ [1].

It is well known that (1) is valid for $n = 2$ [2]. The conjecture is known to be true for the subclass $S_{\mathbf{R}}$ of functions in S with real coefficients. In [1], Brown and Tsao proved that (1) holds for the class T of typically real functions and the class S^* of starlike functions. An analytic function $f(z) = z + \cdots$ is typically real if and only if $(\operatorname{Im} z)(\operatorname{Im} f(z)) \geq 0$ ($z \in D$). And $f(z) = z + \cdots$ is starlike means that $f(D)$ is starlike with respect to the origin.

In this paper, we prove the Zalcman conjecture for the class C of close-to-convex functions and $n \geq 4$. Thus the result in [1] on the starlike functions is an immediate consequence of ours when $n \geq 4$. This provides further evidence in favour of the Zalcman conjecture.

A function $f(z) = z + a_2z^2 + \cdots$ analytic in D is close-to-convex if and only if there exist a real number θ and a function g in S^* such that

$$(2) \quad \operatorname{Re}(e^{i\theta} z f'(z)/g(z)) > 0 \quad (z \in D).$$

We denote by HC the closed convex hull of C .

Our principal result can now be stated.

THEOREM. *Let $f(z) = z + a_2z^2 + a_3a^3 + \cdots \in HC$. Then*

$$(3) \quad |a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (n = 4, 5, \dots).$$

This inequality is sharp.

The proof of the Theorem relies on the following Lemma.

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LEMMA. *Let*

$$(4) \quad F_n(x, y) = (n^2 - 6n + 1)x^2 - 2(n-1)^2xy + 4(n-1)\sqrt{1-x^2}\sqrt{1-y^2} + (n-1)^2y^2,$$

where $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ and $n = 4, 5, \dots$. Then

$$(5) \quad F_n(x, y) \leq 4(n-1)^2 - 4n.$$

The equality holds if and only if $(x, y) = (1, -1)$ or $(x, y) = (-1, 1)$.

PROOF. By applying the inequality $2ab \leq a^2 + b^2$, we have

$$(6) \quad F_n(x, y) \leq 4(n-1) + G_n(x, y),$$

where

$$(7) \quad G_n(x, y) = (n^2 - 8n + 3)x^2 - 2(n-1)^2xy + (n^2 - 4n + 3)y^2.$$

If $n \geq 8$, it is clear that

$$(8) \quad G_n(x, y) \leq 4n^2 - 16n + 8,$$

which implies (5). The equality holds in (8) if and only if $(x, y) = (1, -1)$ or $(x, y) = (-1, 1)$, so does the equality in (5) since $1 - x^2 = 1 - y^2$ at $(1, -1)$ and $(-1, 1)$.

It is not difficult to show that

$$G_5(x, y) = -12x^2 - 32xy + 8y^2, \quad G_6(x, y) = -9x^2 - 50xy + 15y^2,$$

and

$$G_7(x, y) = -4x^2 - 72xy + 24y^2$$

maximize only at $(1, -1)$ and $(-1, 1)$. This gives the required result (5) for $n = 5, 6, 7$.

From (4), we get for $-1 < x < 1$ and $-1 < y < 1$

$$\partial F_4(x, y) / \partial x = -14x - 18y - 12x\sqrt{(1-y^2)/(1-x^2)},$$

$$\partial F_4(x, y) / \partial y = -18x + 18y - 12y\sqrt{(1-x^2)/(1-y^2)}.$$

Both of them vanish only if $(x+y)(7x-9y) = 0$. For $x = -y$ and $-1 < y < 1$,

$$G_4(x, y) = -13x^2 - 18xy + 3y^2 = 8y^2 < 8.$$

For $x = 9y/7$ and $-7/9 < y < 7/9$,

$$G_4(x, y) = -2040y^2/49 < 8.$$

Thus $F_4(x, y) \leq 12 + G_4(x, y) < 20$ if $-1 < x < 1$ and $-1 < y < 1$. It is easy to check that $F_4(x, y)$ maximizes on the boundary of $\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ only at $(1, -1)$ and $(-1, 1)$. This completes the proof of our lemma.

PROOF OF THE THEOREM. Since $|a_n^2 - a_{2n-1}|$ is invariant under rotations, it is sufficient to prove that

$$(9) \quad \operatorname{Re}(a_n^2 - a_{2n-1}) \leq (n-1)^2.$$

It is well known [3] that HC consists of the functions represented by

$$(10) \quad f(z) = \int_R \left(z - \frac{1}{2}(e^{is} + e^{it})z^2 \right) (1 - e^{it}z)^{-2} d\mu(t, s),$$

where μ varies over the set of probability measures on $R = [0, 2\pi] \times [0, 2\pi]$. Thus we have

$$\begin{aligned} a_n &= \int_R \left\{ ne^{i(n-1)t} - \frac{1}{2}(e^{is} + e^{it})(n-1)e^{i(n-2)t} \right\} d\mu \\ &= \frac{1}{2} \int_R \{ (n+1)e^{i(n-1)t} - (n-1)e^{i(s+(n-2)t)} \} d\mu, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re}(a_n^2 - a_{2n-1}) &= \frac{1}{4} \left\{ \int_R [(n+1) \cos(n-1)t - (n-1) \cos(s + (n-2)t)] d\mu \right\}^2 \\ &\quad - \frac{1}{4} \left\{ \int_R [(n+1) \sin(n-1)t - (n-1) \sin(s + (n-2)t)] d\mu \right\}^2 \\ &\quad - \int_R [n \cos 2(n-1)t - (n-1) \cos(s + (2n-3)t)] d\mu. \end{aligned}$$

By applying the Cauchy-Schwarz inequality to

$$\left\{ \int_R [(n+1) \cos(n-1)t - (n-1) \cos(s + (n-2)t)] d\mu \right\}^2,$$

we get

$$\begin{aligned} (11) \quad &\operatorname{Re}(a_n^2 - a_{2n-1}) \\ &\leq \frac{1}{4} \int_R \{ (n+1)^2 \cos^2(n-1)t \\ &\quad - 2(n+1)(n-1) \cos(n-1)t \cos(s + (n-2)t) \\ &\quad \quad + (n-1)^2 \cos^2(s + (n-2)t) \} d\mu \\ &\quad - \int_R \{ n \cos 2(n-1)t - (-1) \cos(s + (2n-3)t) \} d\mu \\ &\quad + n + \frac{1}{4} \int_R \{ (n^2 - 6n + 1) \cos^2(n-1)t \\ &= \quad - 2(n-1)^2 \cos(n-1)t \cos(s + (n-2)t) \\ &\quad \quad + (n-1)^2 \cos^2(s + (n-2)t) \\ &\quad \quad - 4(n-1) \sin(n-1)t \sin(s + (n-2)t) \} d\mu \\ &\leq n + \frac{1}{4} \int_R F_n(\cos(n-1)t, \cos(s + (n-2)t)) d\mu. \end{aligned}$$

If follows from the Lemma that

$$\operatorname{Re}(a_n^2 - a_{2n-1}) \leq n + \frac{1}{4} \int_R (4(n-1)^2 - 4n) d\mu = (n-1)^2,$$

and if the equality holds, we must have

$$\cos(n-1)t = 1 \quad \text{and} \quad \cos(s + (n-2)t) = -1,$$

or

$$\cos(n-1)t = -1 \quad \text{and} \quad \cos(s + (n-2)t) = 1.$$

Moreover, the support of μ is contained in E_1 or E_2 , where

$$E_1 = \{(t, s): t = 2k\pi/(n-1), s = (2m-1)\pi - 2(n-2)k\pi/(n-1), \\ k = 0, 1, \dots, n-1, \text{ and } m \text{ is an integer in } [\frac{1}{2} + (n-2)k/(n-1), 3/2 + (n-2)k/(n-1)]\},$$

$$E_2 = \{(t, s): t = (2k+1)\pi/(n-1), s = 2m\pi - (2k+1)(n-2)\pi/(n-1), \\ k = 0, 1, \dots, n-2, \text{ and } m \text{ is an integer in } [\frac{1}{2}(n-2)(2k+1)/(n-1), 1 + \frac{1}{2}(n-2)(2k+1)/(n-1)]\}.$$

Because $\sin(n-1)t$ and $\sin(s + (n-2)t)$ vanish on E_1 and E_2 , (11) is sharp in HC , and so is (3). We complete the proof of the Theorem.

It is well known that $S^* \subset C \subset HC$. If $f(z) \in C$ with the form (10) and μ has its support in E_i ($i = 1, 2$), then the univalence of $f(z)$ implies that $f(z)$ has only one pole of order two on $|z| = 1$, from which we can show that

$$(12) \quad f(z) = (z - \frac{1}{2}(e^{is} + e^{it})z^2)/(1 - e^{it}z)^2,$$

where (t, s) is in E_i . Therefore we have the following corollary.

COROLLARY 1. *If $f(z) = z + a_2z^2 + \dots \in C$, then*

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (n = 4, 5, \dots).$$

The equality holds if and only if $f(z)$ has the form of (12).

Some of the functions with form (12) are in S^* , so we have the following result given in [1].

COROLLARY 2. *If $f(z) = z + a_2z^2 + \dots \in S^*$, then*

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (n = 4, 5, \dots).$$

This inequality is sharp.

REMARKS. In [1], the authors pointed out that the Zalcman conjecture is false in HS^* for $n = 2$ and 3, so it is also false in HC for $n = 2$ and 3. For $n = 2$, the conjecture is true in S , so also in C . However, the conjecture remains open in C for $n = 3$.

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