AN ELEMENTARY PROOF OF A RESULT ON $\Lambda(p)$ SETS

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(Communicated by William J. Davis)

ABSTRACT. We give an elementary proof of the fact that a subset of the dual of a compact abelian group which is a $\Lambda(p)$ set for some $1 \leq p < 2$ is a $\Lambda(p + \varepsilon)$ set for some $\varepsilon > 0$, and extend this result to $\Lambda(p)$ sets for $0 < p < 1$.

Let G be a compact abelian group with normalized Haar measure m , and let $\hat{G}$ be its discrete dual group. As in [6] we say that $E \subset \hat{G}$ is a $\Lambda(p)$ set for some $p > 0$ provided there exists $0 < q < p$ and a constant $c = c(p, q, E)$ such that

$$(*) \quad \|f\|_p \equiv \left(\int |f|^p dm \right)^{1/p} \leq c \|f\|_q$$

whenever $f \in \text{Trig}_E G \equiv \{f: G \rightarrow \mathbb{C}: \text{supp } \hat{f} \text{ is a finite subset of } E\}$. The least such constant c is called the $\Lambda(p, q)$ constant of E and is denoted by $\Lambda(p, q, E)$. An application of Hölder's inequality shows that if $(*)$ holds for some $q < p$ then it holds for all such q , with different values for c . Moreover, if E is a $\Lambda(p)$ set and $r < p$, then E is a $\Lambda(r)$ set. For standard results on $\Lambda(p)$ sets we refer the reader to [6 or 3].

In [1] Bachelis and Ebenstein showed that if E is a $\Lambda(p)$ set for some $1 \leq p < 2$, then E is a $\Lambda(p + \varepsilon)$ set for some $\varepsilon > 0$. Their proof was based on a deep result by Rosenthal [5]. In this note we give an elementary, self-contained proof of this fact about $\Lambda(p)$ sets and extend it to $\Lambda(p)$ sets with $0 < p < 1$.

THEOREM. Let $0 < p < 2$ and suppose $E \subset \hat{G}$ is a $\Lambda(p)$ set. Then E is a $\Lambda(q)$ set for some $q > p$.

For the proof we need the following three lemmas.

LEMMA 1. Suppose E is a $\Lambda(p)$ set, $p > 0$, with $\Lambda(p, p/2, E) \leq c$. Let $n \geq 4c^p$. Then for every nonzero $f \in \text{Trig}_E G$ and for every $A \subset G$ with $m(A) \leq 1/n$,

$$\int_A |f|^p dm < \left(1 - \frac{1}{n}\right) \|f\|_p^p.$$

PROOF. Suppose there is some nonzero $f \in \text{Trig}_E G$ and $A \subset G$ with $m(A) \leq 1/n$ such that

$$\int_A |f|^p \geq \left(1 - \frac{1}{n}\right) \|f\|_p^p.$$

Received by the editors September 3, 1987 and, in revised form, January 18, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A46.

Research partially supported by NSERC Grant A9333.

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Without loss of generality we may assume $\|f\|_p = 1$. By Hölder's inequality

$$\int_A |f|^{p/2} \leq m(A)^{1/2} \|f\|_p^{p/2} \leq \frac{1}{\sqrt{n}}.$$

Also

$$\int_{A^c} |f|^{p/2} \leq \left(\int_{A^c} |f|^p \right)^{1/2} < \frac{1}{\sqrt{n}}.$$

Hence

$$\|f\|_{p/2} < (2/\sqrt{n})^{2/p} \|f\|_p \leq (1/c) \|f\|_p,$$

which contradicts the assumption that $\Lambda(p, p/2, E) \leq c$. $\square$

Let $p, q, \lambda > 0$ and $n \in \mathbf{Z}$. We will say that $E \subset \hat{G}$ has *property* $P(p, q, \lambda, n)$ if there are $f_1, \dots, f_n \in \text{Trig}_E G$ such that

- (1) $\|f_i\|_p = 1$ for $i = 1, \dots, n$, and
- (2) $\|\sum_{i=1}^n \alpha_i f_i\|_p \geq \lambda (\sum_{i=1}^n |\alpha_i|^q)^{1/q}$ for all scalars $\alpha_1, \dots, \alpha_n$.

LEMMA 2. Suppose $\lambda, \delta > 0$, $0 < p < 2$ and $n \in \mathbf{Z}$. There is an $N = N(\lambda, \delta, p, n)$ so that if E has property $P(p, p, \lambda, N)$, then there are disjoint measurable sets $A_1, \dots, A_n \subset G$ and nonzero $f_1, \dots, f_n \in \text{Trig}_E G$ such that

$$\int_{A_i} |f_i|^p \geq \frac{\lambda^{2p/(2-p)}}{1+\delta} \|f_i\|_p^p$$

for all $i = 1, \dots, n$.

REMARKS. 1. It is proven in [2] and [4] that if E has property $P(p, p, \lambda, N)$ then there are disjoint measurable sets $A_1, \dots, A_N \subset G$ and nonzero $f_1, \dots, f_N \in \text{Trig}_E G$ such that $\int_{A_i} |f_i|^p \geq \lambda^{2p/(2-p)} \|f_i\|_p^p$ for $i = 1, \dots, N$. The proof of this stronger result is not as elementary as the one we give for Lemma 2.

2. The first part of the proof we give below is from [2]. We present it here for completeness. The second part is a modification of a known argument (cf. [7, VII]).

PROOF. Suppose $f_1, \dots, f_N \in \text{Trig}_E G$ and satisfy (1) and (2). Let $r_i(t)$ be the Rademacher functions. By (2), averaging over t using Fubini's theorem, and applying Hölder's inequality, we obtain

$$\begin{aligned} \lambda^p N &\leq \int_0^1 \left\| \sum_{i=1}^N r_i(t) f_i(x) \right\|_p^p dt = \int_0^1 \int_0^1 \left| \sum_{i=1}^N r_i(t) f_i(x) \right|^p dt dx \\ &\leq \int_0^1 \left(\int_0^1 \left| \sum_{i=1}^N r_i(t) f_i(x) \right|^2 dt \right)^{p/2} dx = \int_0^1 \left(\sum_{i=1}^N |f_i(x)|^2 \right)^{p/2} dx \\ &\leq \int_0^1 \left(\max_{1 \leq i \leq N} |f_i(x)|^{2-p} \right)^{p/2} \left(\sum_{i=1}^N |f_i(x)|^p \right)^{p/2} dx \\ &\leq \left(\int_0^1 \left(\max_{1 \leq i \leq N} |f_i(x)|^p dx \right) \right)^{(2-p)/2} \left(\int_0^1 \left(\sum_{i=1}^N |f_i(x)|^p \right) dx \right)^{p/2} \\ &= N^{p/2} \left(\int_0^1 \max_i |f_i(x)|^p dx \right)^{(2-p)/2}. \end{aligned}$$

Thus

$$\int_0^1 \max_{1 \leq i \leq N} |f_i(x)|^p dx \geq \lambda^{2p/(2-p)} N.$$

For $i = 1, \dots, N$ let

$$A_i = \left\{ x: i \text{ is the first index } j \text{ such that } |f_j(x)| = \max_{1 \leq k \leq N} |f_k(x)| \right\}.$$

Obviously the A_i 's are disjoint. Moreover,

$$\lambda^{2p/(2-p)} N \leq \sum_{i=1}^N \int_{A_i} \max_{1 \leq j \leq N} |f_j(x)|^p dx = \sum_{i=1}^N \int_{A_i} |f_i|^p dx.$$

Set

$$J = \left\{ i: \int_{A_i} |f_i|^p dx \geq \frac{\lambda^{2p/(2-p)}}{1+\delta} \right\}.$$

Then

$$\lambda^{2p/(2-p)} N \leq |J| + \frac{\lambda^{2p/(2-p)} N}{1+\delta}.$$

So $|J| \geq N \lambda^{2p/(2-p)} \delta / (1+\delta) \geq n$ if N is sufficiently large. $\square$

For part of the proof of Lemma 3 we use arguments similar to some found in [5, Lemma 6].

LEMMA 3. *Let $0 < p < q$. If E is not a $\Lambda(q)$ set then E has property $P(p, q, \lambda, n)$ for every $0 < \lambda < 1$ and integer n .*

PROOF. Fix $0 < \lambda < 1$ and $n \in \mathbf{Z}$. If $q > 1$ let $q' = q/(q-1)$ and choose $0 < \delta < 1$ so that $\delta - n^{1/q'}(1-\delta^q)^{1/q} \geq \lambda$. For $q \leq 1$ choose $0 < \delta < 1$ satisfying $2\delta^q - 1 \geq \lambda^q$.

Since E is not a $\Lambda(q)$ set we may choose a finite subset F of E with $\Lambda(q, p, F) \equiv M > ((n+1)/(1-\delta^q))^{1/p}$. Select $f_1 \in \text{Trig}_F G$ so that $\|f_1\|_p = 1$ and $\|f_1\|_q = M$. Let $A_1 = \{x: |f_1(x)| > M\}$. Then $m(A_1) \leq M^{-p}$,

$$\int_{A_1^c} |f_1|^q \leq M^{q-p} \int_{A_1^c} |f_1|^p \leq M^{q-p}$$

and

$$\int_{A_1} |f_1|^q \geq M^q - M^{q-p} > \delta^q M^q.$$

(The final inequality is true because of the choice of M .)

Now assume inductively that for some $1 \leq k \leq n$ there exist $f_1, \dots, f_k \in \text{Trig}_F G$ and disjoint measurable sets $A_1, \dots, A_k \subset G$, with each f_i a translate of f_1 , $m(A_i) \leq M^{-p}$, and $\|f_i 1_{A_i}\|_q > \delta M$.

Let $B = \bigcup_1^k A_i$. Then $m(B^c) \geq 1 - kM^{-p} \geq 1 - nM^{-p}$. Also, if we let $f_g(x) = f_1(x - g)$, then Fubini's theorem shows that

$$\int_G \int_{B^c} |f_g(x)|^q = \int_{B^c} \|f_1\|_q^q \geq M^q (1 - nM^{-p}).$$

Hence there exists some $g \in G$ such that for $f_{k+1} \equiv f_g$, $\int_{B^c} |f_{k+1}|^q \geq M^q(1 - nM^{-p})$. Set $A_{k+1} = \{x: |f_{k+1}(x)| > M\} \cap B^c$. Then $m(A_{k+1}) \leq M^{-p}$, $A_{k+1} \cap A_i = \emptyset$ for $i \leq k$ and

$$\int_{B^c \setminus A_{k+1}} |f_{k+1}|^q \leq M^{q-p} \int_{B^c \setminus A_{k+1}} |f_{k+1}|^p \leq M^{q-p}.$$

Therefore

$$\int_{A_{k+1}} |f_{k+1}|^q \geq M^q(1 - nM^{-p}) - M^{q-p} > \delta^q M^q$$

which completes the induction step.

Now let $\varphi_i = f_i 1_{A_i}$ for $i = 1, \dots, n$. Then

$$(**) \quad \|\varphi_i - f_i\|_q^q = \int_{A_i^c} |f_i|^q \leq M^q(1 - \delta^q).$$

If $q \leq 1$, (**), together with the triangle inequality, shows that for any choice of scalars $\alpha_1, \dots, \alpha_n$,

$$\left\| \sum_{i=1}^n \alpha_i (\varphi_i - f_i) \right\|_q^q \leq \sum_{i=1}^n |\alpha_i|^q M^q(1 - \delta^q);$$

while if $q > 1$, (**), the triangle inequality and Hölder's inequality combine to give

$$\left\| \sum \alpha_i (\varphi_i - f_i) \right\|_q \leq \left(\sum_{i=1}^n |\alpha_i|^q \right)^{1/q} n^{1/q'} M(1 - \delta^q)^{1/q}.$$

Since $A_1, \dots, A_n$ are disjoint it follows that if $q \leq 1$,

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i f_i \right\|_q^q &\geq \left\| \sum \alpha_i \varphi_i \right\|_q^q - \left\| \sum \alpha_i (\varphi_i - f_i) \right\|_q^q \\ &\geq \sum_{i=1}^n \int_{A_i} |\alpha_i f_i|^q - \sum_{i=1}^n |\alpha_i|^q M^q(1 - \delta^q) \\ &\geq \sum_{i=1}^n |\alpha_i|^q M^q(\delta^q - (1 - \delta^q)) \\ &= \sum_{i=1}^n |\alpha_i|^q M^q \lambda^q; \end{aligned}$$

while if $q > 1$,

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i f_i \right\|_q &\geq \left\| \sum \alpha_i \varphi_i \right\|_q - \left\| \sum \alpha_i (\varphi_i - f_i) \right\|_q \\ &\geq \left(\sum_i \int_{A_i} |\alpha_i f_i|^q \right)^{1/q} - \left(\sum |\alpha_i|^q \right)^{1/q} n^{1/q'} M(1 - \delta^q)^{1/q} \\ &\geq \left(\sum_i |\alpha_i|^q \right)^{1/q} M(\delta - n^{1/q'}(1 - \delta^q)^{1/q}) \\ &\geq \left(\sum_i |\alpha_i|^q \right)^{1/q} M\lambda. \end{aligned}$$

But the choice of F ensures that

$$\left\| \sum \alpha_i f_i \right\|_p \geq \frac{1}{M} \left\| \sum \alpha_i f_i \right\|_q,$$

hence in either case

$$\left\| \sum_{i=1}^n \alpha_i f_i \right\|_p \geq \lambda \left(\sum_{i=1}^n |\alpha_i|^q \right)^{1/q}$$

for all scalars $\alpha_1, \dots, \alpha_n$ and so E has property $P(p, q, \lambda, n)$. $\square$

PROOF OF THE THEOREM. Suppose E is a $\Lambda(p)$ set with $c = \Lambda(p, p/2, E)$. Let n be an integer with $n \geq 4c^p$, let $\lambda = (1 - 1/n^2)^{(2-p)/2p}$ and let $N = N(\lambda, 1/n, p, n)$ as in Lemma 2.

If E has property $P(p, p, \lambda, N)$ then by Lemma 2 there are disjoint measurable sets $A_1, \dots, A_n \subset G$ and nonzero $f_1, \dots, f_n \in \text{Trig}_E G$ such that

$$\int_{A_i} |f_i|^p \geq \left(\frac{1 - 1/n^2}{1 + 1/n} \right) \|f_i\|_p^p = (1 - 1/n) \|f_i\|_p^p$$

for all $i = 1, \dots, n$. For some i , $m(A_i) \leq 1/n$ and this contradicts Lemma 1. Thus E cannot have property $P(p, p, \lambda, N)$.

Let $q > p$ satisfy $N^{1/q-1/p} \geq \sqrt{\lambda}$ and suppose E has property $P(p, q, \sqrt{\lambda}, N)$. Assume $f_1, \dots, f_N \in \text{Trig}_E G$ satisfy

- (1) $\|f_i\|_p = 1$ for $i = 1, \dots, N$, and
- (2) $\left\| \sum_{i=1}^N \alpha_i f_i \right\|_p \geq \sqrt{\lambda} \left(\sum_{i=1}^N |\alpha_i|^q \right)^{1/q}$ for all scalars $\alpha_1, \dots, \alpha_N$.

By Hölder's inequality

$$\left(\sum_{i=1}^N |\alpha_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^N |\alpha_i|^q \right)^{1/q} N^{1/p-1/q},$$

thus

$$\begin{aligned} \left\| \sum_{i=1}^N \alpha_i f_i \right\|_p &\geq N^{1/q-1/p} \sqrt{\lambda} \left(\sum_{i=1}^N |\alpha_i|^p \right)^{1/p} \\ &\geq \lambda \left(\sum_{i=1}^N |\alpha_i|^p \right)^{1/p}. \end{aligned}$$

This means that E has property $P(p, p, \lambda, N)$ which contradicts our work above. Hence E fails to have property $P(p, q, \sqrt{\lambda}, N)$, and so by Lemma 3 is a $\Lambda(q)$ set. $\square$

REMARK. Our method of proof can be used to show that if E is a $\Lambda(p)$ set for some $0 < p < 2$, with $\Lambda(p, p/2, E) \leq c$, then E is a $\Lambda(q)$ set with

$$\Lambda(q, p, E) \leq \begin{cases} (\frac{32pn^4}{2-p})^{q/p} & \text{if } p \geq 1 \\ (\frac{32n^4}{2-p})^{1/p} & \text{if } p < 1, \end{cases}$$

where $n \in \mathbb{Z}$, $n \geq 4c^p$ and

$$\frac{1}{p} > \frac{1}{q} \geq \frac{1}{p} + \frac{2-p \log(1-1/n^2)}{4p \log 2n^2}.$$

COROLLARY. L_E^1 is reflexive if and only if E is a $\Lambda(1)$ set.

PROOF. If E is a $\Lambda(1)$ set then E is a $\Lambda(p)$ set for some $p > 1$. Hence $L_E^1 = L_E^p$ and thus is reflexive.

If L_E^1 is reflexive then the unit ball of L_E^1 is uniformly integrable. Hence there exists $\delta > 0$ so that whenever $m(A) < \delta$ and $f \in L_E^1$ then $\int_A |f| \leq \frac{1}{2} \|f\|_1$.

Let $f \in \text{Trig}_E G$ with $\|f\|_1 = 1$. Let $A = \{x: |f(x)| > 1/\delta\}$. Then $m(A) < \delta$ so $\int_A |f| \leq \frac{1}{2}$ and hence $\int_{A^c} |f| \geq \frac{1}{2}$. Thus

$$\|f\|_{1/2} \geq \left(\int_{A^c} |f|^{1/2} \right)^2 \geq \left(\sqrt{\delta} \int_{A^c} |f| \right)^2 \geq \frac{\delta}{4} \|f\|_1$$

and since f was arbitrary E is a $\Lambda(1)$ set. $\square$

The author would like to thank Dr. Tomczak-Jaegermann for a number of helpful discussions.

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