THE FIRST DIRICHLET EIGENVALUE AND RADIUS OF A GEODESIC BALL

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ABSTRACT. We give certain relation between the first Dirichlet eigenvalue and radius of a geodesic ball in a connected, compact n -dimensional Riemannian globally symmetric space of rank one.

Let M be a connected, compact n -dimensional Riemannian globally symmetric space of rank one, L the diameter of M , $A(r)$ the surface area of a sphere with radius r ($0 < r < L$), in M , Δ the Laplace-Beltrami operator on M , and $B(r)$ denote a geodesic ball with radius r , in M .

Consider the following Dirichlet eigenvalue problem:

$$\Delta f + \lambda f = 0 \text{ in } B(r), \quad f = 0 \text{ on } \partial B(r).$$

Let $\lambda = \lambda(M, n, r)$ be the first Dirichlet eigenvalue of $B(r)$. By domain monotonicity of Dirichlet eigenvalues, for fixed n , $\lambda(M, n, r): (0, L) \rightarrow (0, \infty)$ is strictly decreasing. For our convenience, we introduce $r = r(M, n, \lambda)$ which is the inverse function of $\lambda(M, n, r)$.

THEOREM. *If $\lambda(M, n, r) = 2k\alpha(n+2\beta+2k)$ ($k = 1, 2, 3, \dots$), then $r = r(M, n, \lambda)$ is the first positive zero of*

$$T(t) = 1 + \sum_{j=1}^k (-\sin^2 \sqrt{\alpha} t)^j \prod_{m=0}^{j-1} \frac{(k-m)(n+2\beta+2k+2m)}{(m+1)(n+2m)},$$

where α and β are constants determined by M . (See Table 1.) In particular,

$$\begin{aligned} \sqrt{\alpha} r(M, n, 2\alpha(n+2\beta+2)) &= \arcsin \sqrt{n/(n+2\beta+2)}, \\ \sqrt{\alpha} r(M, n, 4\alpha(n+2\beta+4)) &= \arcsin \sqrt{\{n+2 - \sqrt{(4\beta+8)(n+2)/(n+2\beta+4)}\}/(n+2\beta+6)}. \end{aligned}$$

PROOF OF THE THEOREM. It is well known that M is isometric to one of the following spaces: $S^m(\alpha)$ (the sphere with constant curvature α), $P^m(\alpha)$ (the real projective space with constant curvature α), $CP^m(\alpha)$ (the complex projective space with constant holomorphic sectional curvature 4α), $QP^m(\alpha)$ (the quaternionic space with maximum sectional curvature 4α), and $\text{Cay } P^2(\alpha)$ (the Cayley plane with maximum sectional curvature 4α), and also well known that

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$A(r) = C(\sin \sqrt{\alpha} r)^{n-1}(\cos \sqrt{\alpha} r)^{2\beta+1}$ for some constant C determined by M . (See [3].)

M	L	β	n
$S^m(\alpha)$	$\pi/\sqrt{\alpha}$	$-1/2$	m
$P^m(\alpha)$	$\pi/2\sqrt{\alpha}$	$-1/2$	m
$CP^m(\alpha)$	$\pi/2\sqrt{\alpha}$	0	$2m$
$QP^m(\alpha)$	$\pi/2\sqrt{\alpha}$	1	$4m$
$\text{Cay } P^2(\alpha)$	$\pi/2\sqrt{\alpha}$	3	16

TABLE 1

Note that we may take $\alpha = 1$. Since M is two-point homogeneous, the eigenfunction of $\lambda(M, n, r)$ is radial. In geodesic polar coordinates on M , the radial part of Δ is

$$\Delta_t = \frac{d^2}{dt^2} + \left(\frac{dA}{dt} / A(t) \right) \frac{d}{dt} \quad (0 < t < L).$$

Now the eigenfunction $T(t)$ of $\lambda(M, n, r)$ satisfies

$$T'' + [(n-1) \cot t - (2\beta+1) \tan t] T'(t) + \lambda T(t) = 0.$$

Solving ODE we have

$$T(t) = F(a, b; n/2; \sin^2 t) \quad (t \leq r < \pi/2),$$

where F is the Gauss hypergeometric function, $a = (n+2\beta + \sqrt{(n+2\beta)^2 + 4\lambda})/4$, and $b = (n+2\beta - \sqrt{(n+2\beta)^2 + 4\lambda})/4$. Finally, we obtain

$$T(t) = 1 + \sum_{j=1}^{\infty} (\sin^2 t/2)^j \prod_{m=1}^j \frac{-\lambda + 2(m-1)(n+2\beta+2m-2)}{m(n+2m-2)} \quad (t \leq r < \pi/2).$$

Since $T(t)$ does not vanish in $B(r)$, $r = r(M, n, \lambda)$ is the first positive zero of $T(t)$. This completes the proof. $\square$

REMARK. When $M = S^m(\alpha)$ or $P^m(\alpha)$, using the properties of the Gauss hypergeometric functions, we have many results of different type which are not available for other spaces $CP^m(\alpha)$, $QP^m(\alpha)$ and $\text{Cay } P^2(\alpha)$.

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