POSITIVELY EXPANSIVE MAPS AND GROWTH OF FUNDAMENTAL GROUPS

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ABSTRACT. In this note we will show that a positively expansive map of an arbitrary closed topological manifold is topologically conjugate to an expanding infra-nil-endomorphism.

Let Γ be a group generated by $\{\gamma_1, \dots, \gamma_k\}$. Then each $\gamma \in \Gamma$ is represented as a word $\gamma_{i_1}^{p_1} \gamma_{i_2}^{p_2} \dots \gamma_{i_l}^{p_l}$ and the number $|p_1| + |p_2| + \dots + |p_l|$ is called the *length* of the word. The *norm* $\|\gamma\|$ is defined as the minimal length of the word representing γ . As properties of the norm we know that $\|\gamma\| = \|\gamma^{-1}\|$ and $\|\gamma\gamma'\| \leq \|\gamma\| + \|\gamma'\|$. If $\delta_1, \dots, \delta_k$ is another system of generators in Γ , then the corresponding norm $\|\cdot\|'$ is not necessarily equal to $\|\cdot\|$. But there exists $c > 0$ such that for each $\gamma \in \Gamma$

$$c\|\gamma\|' \geq \|\gamma\| \geq c^{-1}\|\gamma\|'.$$

Let Γ be a finitely generated group and fix its generators. We denote by $B(t)$, $t > 0$, the ball of radius t centered at the identity element e , and by $\#B(t)$ the number of elements in $B(t)$.

We say that Γ has *polynomial growth* if there are two positive numbers d and c such that for all balls $B(t)$, $t \geq 1$, $\#B(t) \leq ct^d$. If Γ has a nilpotent subgroup of finite index then Γ has polynomial growth (Wolf [10]).

The following is one interesting result for our investigation.

THEOREM A (M. GROMOV [5]). *If a finitely generated group Γ has polynomial growth, then Γ contains a nilpotent subgroup of finite index.*

Let L be a simply connected nilpotent Lie group with a left invariant Riemannian metric and denote by $\text{Aff}(L)$ the group of transformations of L generated by the left translations and by all automorphisms from L onto itself. Let $\Gamma \subset \text{Aff}(L)$ be a group which acts freely and discretely on L . If the quotient $X = L/\Gamma$ is compact, then it is called an *infra-nil-manifold*. Each expanding automorphism $L \rightarrow L$ which respects Γ induces an expanding map $X \rightarrow X$. Such maps are called an *expanding infra-nil-endomorphism*.

From the following results together with Theorem A, M. Gromov [5] proved that an expanding differentiable map of an arbitrary closed C^∞ manifold is topologically conjugate to an infra-nil-endomorphism.

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THEOREM B (M. SHUB [9]). *An expanding differentiable map of a closed C^∞ manifold M is topologically conjugate to an expanding infra-nil-endomorphism if and only if the fundamental group $\pi_1(M)$ contains a nilpotent subgroup of finite index.*

THEOREM C (J. FRANKS [4]). *If a closed C^∞ manifold M admits an expanding differentiable map, then the fundamental group $\pi_1(M)$ has polynomial growth.*

Our aim is to prove that Gromov's result is true for positively expansive maps being more general than expanding differentiable maps.

A continuous map $f: X \rightarrow X$ of a metric space (X, d) is called *positively expansive* if there is a constant $c > 0$ such that if $x \neq y$ then $d(f^n(x), f^n(y)) \geq c$ for some $n \geq 0$. If, in particular, X is a closed topological manifold and f is positively expansive, then f is a covering map with covering degree ≥ 2 .

Our assertion is the following

THEOREM. *A positively expansive map of an arbitrary closed topological manifold is topologically conjugate to an expanding infra-nil-endomorphism.*

Hereafter, let M be a closed topological manifold and $\pi: \bar{M} \rightarrow M$ be the universal cover. For the proof of our theorem we need the following

LEMMA 1 (E. COVEN AND W. REDDY [2]). *If $f: M \rightarrow M$ is a positively expansive map, then there exists a metric $\bar{d}$ for $\bar{M}$ such that*

- (i) $\bar{d}$ is complete,
- (ii) all covering transformations are isometries,
- (iii) if $\bar{f}: \bar{M} \rightarrow \bar{M}$ is a lift of f by π , then there is $\lambda > 1$ such that $\bar{d}(\bar{f}(x), \bar{f}(y)) \geq \lambda \bar{d}(x, y)$ for $x, y \in \bar{M}$.

By making use of Shub [8] and Lemma 1, (I) and (II) below are easily obtained (see P. Duvall and L. Husch [3] for the proof of (I)(a) and refer R. Kirby and L. Siebenmann [6] for the proof of (II)):

- (I) [8, pp. 171–181] If $f: M \rightarrow M$ is a positively expansive map, then
 - (a) $\bar{M}$ is homeomorphic to $\mathbb{R}^n$ ($n = \dim(\bar{M})$),
 - (b) f has a fixed point,
 - (c) the nonwandering set of f , $\Omega(f)$, coincides with the whole space (i.e. $\Omega(f) = M$),
 - (d) $\pi_1(M)$ is torsion free,
 - (e) M is of type $K(\pi, 1)$.

(II) [8, pp. 181–184] Let $g: N \rightarrow N$ be a continuous map of a compact topological manifold N and have a fixed point n_0 , and let m_0 be a fixed point of a positively expansive map $f: M \rightarrow M$. If there is a group homomorphism $A: \pi_1(N, n_0) \rightarrow \pi_1(M, m_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(N, n_0) & \xrightarrow{g_*} & \pi_1(N, n_0) \\ A \downarrow & & \downarrow A \\ \pi_1(M, m_0) & \xrightarrow{f_*} & \pi_1(M, m_0) \end{array}$$

then one finds a unique continuous map $h: N \rightarrow M$ such that $h(n_0) = m_0$, $h_* = A$ and $h \circ g = f \circ h$.

PROPOSITION 2. *If M admits a positively expansive map, then the fundamental group $\pi_1(M)$ has polynomial growth.*

If we establish Proposition 2, then our theorem is proved as follows. Let $f: M \rightarrow M$ be a positively expansive map. Then by (I)(b), f has a fixed point m_0 . By Theorem A, $\pi_1(M)$ contains a nilpotent subgroup of finite index. Since $f: M \rightarrow M$ is a covering map and $\pi_1(M)$ is torsion free (by (I)(d)), by Franks [4, Existence Theorem 3.5], there exist an infra-nil-manifold L/Γ , an infra-nil-endomorphism $g: L/\Gamma \rightarrow L/\Gamma$ and an isomorphism $A: \pi_1(L/\Gamma, e) \rightarrow \pi_1(M, m_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(L/\Gamma, e) & \xrightarrow{g_*} & \pi_1(L/\Gamma, e) \\ A \downarrow & & \downarrow A \\ \pi_1(M, m_0) & \xrightarrow{f_*} & \pi_1(M, m_0) \end{array}$$

Then by (II) we can find a continuous map $h: L/\Gamma \rightarrow M$ with $h(e) = m_0$ such that $h_* = A$ and $h \circ g = f \circ h$. Combining this with Lemma 1, by the technique of J. Franks [4, p. 91] we see that g is expanding. Hence there is a continuous map $h': M \rightarrow L/\Gamma$ with $h'(m_0) = e$ such that $h'_* = A^{-1}$ and $h' \circ f = g \circ h'$ [8, Theorem 4]. Since $h \circ g = f \circ h$, it follows that $h \circ h' \circ f = f \circ h \circ h'$ and $g \circ h' \circ h = h' \circ h \circ g$. By uniqueness we obtain that $h \circ h'$ and $h' \circ h$ are identity maps. The proof of the theorem is completed.

It only remains to prove Proposition 2. To do this, let d be a metric for M . Then we know (cf. [3]) that there are a metric ρ for $\overline{M}$ and $\eta_0 > 0$ such that

- (1) if $\rho(x, y) \leq \eta_0$, then $d(\pi(x), \pi(y)) = \rho(x, y)$,
- (2) all covering transformations are isometries under ρ .

Fix $0 < \eta < \eta_0/2$ and let $x, y \in \overline{M}$. By a η -chain from x to y we mean a finite sequence $(x_0, \dots, x_n)$ of points in $\overline{M}$ such that $x = x_0$, $y = x_n$ and $\rho(x_i, x_{i+1}) \leq \eta$ for $0 \leq i \leq n-1$. Define

$$\tilde{d}(x, y) = \inf \left\{ \sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \right\}$$

where the infimum is taken over all η -chains from x to y . Then by the triangle inequality we have

$$(3) \quad \tilde{d}(x, y) \geq \rho(x, y) \quad \text{for all } x, y \in \overline{M},$$

from which it follows that $\tilde{d}$ is a metric for $\overline{M}$. If $\rho(x, y) \leq \eta$, then by the definition of $\tilde{d}$, $\tilde{d}(x, y) \leq \rho(x, y)$ and hence $\tilde{d}(x, y) = \rho(x, y)$ (by (3)). Therefore

$$(4) \quad \rho(x, y) \leq \eta \Rightarrow \tilde{d}(x, y) = \rho(x, y).$$

By (2) it is easily checked that

- (5) all covering transformations are isometries under $\tilde{d}$.

We can consider that $\pi_1(M)$ is the group of all covering transformations. Let K be a compact covering domain for π and put

$$F = \{\gamma \in \pi_1(M) : \tilde{d}(\gamma(K), K) \leq \eta\}$$

where $\tilde{d}(\gamma(K), K) = \min\{\tilde{d}(x, y) : x \in \gamma(K), y \in K\}$. Then F is finite. For, if not, there is an infinite subset $\{\gamma_i\}_{i \in \mathbb{N}} \subset \pi_1(M)$ such that $\tilde{d}(\gamma_i(K), K) \leq \eta$. Take $y_i \in \gamma_i(K)$ such that $\tilde{d}(y_i, K) \leq \eta$. Since $\gamma_i^{-1}(y_i) \in K$ and K is compact, we may assume that $\gamma_i^{-1}(y_i)$ converges to some $y_\infty \in K$ as $i \rightarrow \infty$. If $\tilde{d}(\gamma_i(y_\infty), \gamma_j(y_\infty)) \leq \eta_0$, then

$$\begin{aligned} \tilde{d}(\gamma_i(y_\infty), \gamma_j(y_\infty)) &\geq \rho(\gamma_i(y_\infty), \gamma_j(y_\infty)) \quad (\text{by (3)}) \\ &= d(\pi \circ \gamma_i(y_\infty), \pi \circ \gamma_j(y_\infty)) \quad (\text{by (1)}) \\ &= d(\pi(y_\infty), \pi(y_\infty)) = 0, \end{aligned}$$

and so $\rho(\gamma_i(y_\infty), \gamma_j(y_\infty)) = 0$, from which $\gamma_i = \gamma_j$. Hence $\tilde{d}(\gamma_i(y_\infty), \gamma_j(y_\infty)) > \eta_0$ for all $i \neq j$. Since

$$\begin{aligned} \tilde{d}(\gamma_i(y_\infty), y_i) &= \tilde{d}(y_\infty, \gamma_i^{-1}(y_i)) \quad (\text{by (5)}) \\ &\rightarrow 0 \quad \text{as } i \rightarrow \infty, \end{aligned}$$

we have $\tilde{d}(y_i, y_j) > \eta_0$ for sufficiently large $i \neq j$. On the other hand, since $\tilde{d}(y_i, K) \leq \eta$, there is $z_i \in K$ such that $\tilde{d}(z_i, y_i) \leq \eta$. Since K is compact, we may assume that z_i converges to some point in K . Then for $\varepsilon > 0$ there is $i_0 > 0$ such that $\tilde{d}(y_i, y_j) \leq 2\eta + \varepsilon$ for $i, j \geq i_0$, which contradicts $2\eta < \eta_0$. Therefore F is finite.

Since $\pi_1(M)$ is finitely generated, we may assume F is a system of generators in $\pi_1(M)$ (choose suitably a finite subset of $\pi_1(M)$ and take a union of it and F if necessary). We write

$$\nu = \inf\{\tilde{d}(\gamma(K), K) : \gamma \in \pi_1(M), \gamma \notin F\}.$$

By the choice of F it follows that $\nu \geq \eta$. For fixed $y_0 \in K$ write

$$\mu = \max\{\tilde{d}(\gamma(y_0), y_0) : \gamma \in F\}.$$

Claim 1. If $\tilde{d}(y_0, \gamma(K)) \leq \nu t$ for $t \geq 0$, then $\|\gamma\| \leq 2t + 2$ (compare with Milnor [7]).

Let $t \geq 0$ and choose $y \in \gamma(K)$ such that $\tilde{d}(y_0, y) \leq \nu t$. By the definition of $\tilde{d}$, for any $\varepsilon > 0$ there is a η -chain $(x'_0, \dots, x'_n)$ from y_0 to y such that

$$\sum_{i=0}^{n-1} \rho(x'_i, x'_{i+1}) - \varepsilon \leq \tilde{d}(y_0, y).$$

From (4) together with $\nu \geq \eta$ it follows that $(x'_0, \dots, x'_n)$ is a ν -chain from y_0 to y with respect to $\tilde{d}$ and

$$\sum_{i=0}^{n-1} \tilde{d}(x'_i, x'_{i+1}) - \varepsilon \leq \tilde{d}(y_0, y).$$

In the case when $\tilde{d}(x'_i, x'_{i+1}) + \dots + \tilde{d}(x'_{i+j-1}, x'_{i+j}) \leq \nu$, let us take $(x'_{i+1}, \dots, x'_{i+j-1})$ away from $(x'_0, \dots, x'_n)$. Then we have a ν -chain $(x_0, \dots, x_m)$ from y_0 to y with respect to $\tilde{d}$ such that

$$\begin{aligned} \sum_{i=0}^{m-1} \tilde{d}(x_i, x_{i+1}) - \varepsilon &\leq \tilde{d}(y_0, y), \\ \nu &< \tilde{d}(x_i, x_{i+1}) + \tilde{d}(x_{i+1}, x_{i+2}) \quad (0 \leq i \leq m-2). \end{aligned}$$

These two inequalities yield

$$\nu(m/2 - 1) \leq \tilde{d}(y_0, y) + \varepsilon,$$

and hence

$$m \leq (2/\nu)\{\tilde{d}(y_0, y) + \varepsilon\} + 2 \leq 2t - 2\varepsilon/\nu + 2.$$

Since K is a covering domain, we can find $h_i \in \pi_1(M)$ such that $x_i \in h_i(K)$ for $0 \leq i \leq m$. Put $h_0 = \text{id}$ and $h_m = \gamma$. Since each h_i is an isometry (by (5)), we have

$$\tilde{d}(h_i^{-1} \circ h_{i+1}(K), K) = \tilde{d}(h_{i+1}(K), h_i(K)) \leq \tilde{d}(x_{i+1}, x_i) \leq \nu.$$

Hence there is $f_i \in F$ such that $h_i^{-1} \circ h_{i+1} = f_i$ ($0 \leq i \leq m-1$), and so

$$\begin{aligned} \gamma &= (h_0^{-1} \circ h_1) \circ (h_1^{-1} \circ h_2) \circ \cdots \circ (h_{m-1}^{-1} \circ h_m) \\ &= f_0 \circ \cdots \circ f_{m-1}, \end{aligned}$$

from which $\|\gamma\| \leq 2t - 2\varepsilon/\nu + 2$. Since ε is arbitrary, Claim 1 holds.

Claim 2. If $\|\gamma\| \leq t$, then $\tilde{d}(y_0, \gamma(y_0)) \leq \mu t$.

Write $\gamma = f_1 \circ f_2 \circ \cdots \circ f_s$ ($s \leq t$) where $f_i \in F$ for $1 \leq i \leq s$. Then we have

$$\begin{aligned} \tilde{d}(y_0, \gamma(y_0)) &\leq \tilde{d}(y_0, f_1(y_0)) + \tilde{d}(f_1(y_0), f_1 \circ f_2(y_0)) \\ &\quad + \cdots + \tilde{d}(f_1 \circ \cdots \circ f_{s-1}(y_0), f_1 \circ \cdots \circ f_s(y_0)) \\ &= \tilde{d}(y_0, f_1(y_0)) + \tilde{d}(y_0, f_2(y_0)) + \cdots + \tilde{d}(y_0, f_s(y_0)) \\ &\leq \mu s \leq \mu t. \end{aligned}$$

Let $\Gamma = \{\gamma(y_0) : \gamma \in \pi_1(M)\}$ and put $\tilde{B}(t) = \Gamma \cap \tilde{B}_t(y_0)$ where $\tilde{B}_t(y_0) = \{y \in \bar{M} : \tilde{d}(y_0, y) \leq t\}$. By Claims 1 and 2 we have

$$\#\tilde{B}(2\mu t + 2\mu) \geq \#B(2t + 2) \geq \#\tilde{B}(\nu t) \quad (t \geq 0)$$

and hence letting $c = \max\{\mu, \nu/2, \mu^{-1}, (\nu/2)^{-1}\}$,

$$(6) \quad \#\tilde{B}(ct) \geq \#B(t) \geq \#\tilde{B}(c^{-1}(t-2)) \quad (t \geq 2).$$

Take a compact subset D and $s > 0$, and define

$$M_s(D) = \max\{\#E : E \in \mathcal{S}(D, s)\}$$

where

$$\mathcal{S}(D, s) = \{E \subset D : x, y \in E (x \neq y) \Rightarrow \tilde{d}(x, y) \geq s\}.$$

Then clearly $M_{s_1}(D) \geq M_{s_2}(D)$ if $s_1 \leq s_2$. Let $t \geq 0$ and $x \in \bar{M}$. Then it is checked that $\tilde{B}_t(x)$ is compact. Indeed, let K be as above and let $N_\eta(K) = \{y \in \bar{M} : \tilde{d}(y, K) \leq \eta\}$. Then by the choice of F , $N_\eta(K) \subset \bigcup_{\gamma \in F} \gamma(K)$. Since F is finite, $N_\eta(K)$ is compact. Note that $N_\eta(K)$ also is a covering domain for π . Then we have $N_{2\eta}(K) = N_\eta(N_\eta(K))$ is compact, and so is $N_{n\eta}(K)$ for all $n \in \mathbb{N}$. Since $\tilde{B}_t(x) \subset N_{n\eta}(K)$ for some n , $\tilde{B}_t(x)$ is compact.

Claim 3. $M_\eta(\tilde{B}_t(y_0)) \geq \#\tilde{B}(t)$ for $t \geq 0$.

Take $x, y \in \tilde{B}(t)$ and let $\tilde{d}(x, y) \leq \eta$. Then $\rho(x, y) \leq \eta$ by (3), and so by (1)

$$\rho(x, y) = d(\pi(x), \pi(y)) = d(\pi(y_0), \pi(y_0)) = 0.$$

Hence $x = y$. This implies $\tilde{B}(t) \in \mathcal{S}(\tilde{B}_t(y_0), \eta)$. Therefore Claim 3 holds.

Let K be as above and let c_1 be the diameter of K . Then we have

Claim 4. $\# \tilde{B}(t + c_1) \geq M_{3c_1}(\tilde{B}_t(y_0))$ for $t \geq 0$.

Choose $E \in \mathcal{S}(\tilde{B}_t(y_0), 3c_1)$ such that $\#E = M_{3c_1}(\tilde{B}_t(y_0))$. For $x, y \in E$ with $x \neq y$ we have $\tilde{B}_{c_1}(x) \cap \tilde{B}_{c_1}(y) = \emptyset$. Since K is a covering domain, there are $\gamma_x, \gamma_y \in \pi_1(M)$ such that $x \in \gamma_x(K)$ and $y \in \gamma_y(K)$. By (5) and the choice of c_1 , $\gamma_x(K) \subset \tilde{B}_{c_1}(x)$ and $\gamma_y(K) \subset \tilde{B}_{c_1}(y)$. Hence $\gamma_x(K) \cap \gamma_y(K) = \emptyset$ and so $\gamma_x(y_0) \neq \gamma_y(y_0)$. Since $\tilde{B}_{c_1}(x)$ and $\tilde{B}_{c_1}(y)$ are contained in $B_{t+c_1}(y_0)$, we have $\gamma_x(y_0), \gamma_y(y_0) \in \tilde{B}(t + c_1)$. The conclusion is obtained from the existence of the injection $E \rightarrow \tilde{B}(t + c_1)$.

Let $\delta = \min\{\eta, 3c_1, \eta^{-1}, (3c_1)^{-1}\}$. Then (6) and Claims 3 and 4 tell us that

$$(7) \quad M_\delta(\tilde{B}_{ct}(y_0)) \geq \#B(t) \geq M_{\delta^{-1}}(\tilde{B}_{c^{-1}(t-2)-c_1}(y_0)) \quad (t \geq cc_1 + 2).$$

Now we are ready to prove Proposition 2. Let $f: M \rightarrow M$ be a positively expansive map and $\bar{d}$ be as in Lemma 1. For $x, y \in M$ define

$$d(x, y) = \min\{\bar{d}(\bar{x}, \bar{y}) : \bar{x} \in \pi^{-1}(x), \bar{y} \in \pi^{-1}(y)\}.$$

Then by Lemma 1(ii) it is easily checked that d is a metric for M and there is $\eta_0 > 0$ such that $\bar{d}(x, y) \leq \eta_0$ implies $d(\pi(x), \pi(y)) = \bar{d}(x, y)$. Hence $\bar{d}$ satisfies properties (1) and (2). Fix $0 < \eta < \eta_0/2$ and define the metric $\tilde{d}$ for $\bar{M}$ as above. Let $\bar{f}: \bar{M} \rightarrow \bar{M}$ be the lift of f by π so that $\bar{f}(y_0) = y_0$ for some $y_0 \in \bar{M}$ (see (I)(b)). Then we can find $\lambda > 1$ such that $\tilde{d}(\bar{f}(x), \bar{f}(y)) \geq \lambda \tilde{d}(x, y)$ for $x, y \in \bar{M}$. Indeed, for $\varepsilon > 0$ there is a η -chain $(x_0, \dots, x_n)$ from $\bar{f}(x)$ to $\bar{f}(y)$ such that

$$\sum_{i=0}^{n-1} \tilde{d}(x_i, x_{i+1}) \leq \tilde{d}(\bar{f}(x), \bar{f}(y)) + \varepsilon.$$

By Lemma 1(iii) there is $\lambda > 1$ such that $\bar{d}(\bar{f}(u), \bar{f}(v)) \geq \lambda \bar{d}(u, v)$ for $u, v \in \bar{M}$. Hence $\bar{d}(\bar{f}^{-1}(x_i), \bar{f}^{-1}(x_{i+1})) \leq \lambda^{-1} \bar{d}(x_i, x_{i+1}) < \eta$, and so $(\bar{f}^{-1}(x_0), \dots, \bar{f}^{-1}(x_n))$ is a η -chain from x to y . By this fact we can calculate that

$$\begin{aligned} \tilde{d}(x, y) &\leq \sum_{i=0}^{n-1} \bar{d}(\bar{f}^{-1}(x_i), \bar{f}^{-1}(x_{i+1})) \\ &\leq \lambda^{-1} \sum_{i=0}^{n-1} \tilde{d}(x_i, x_{i+1}) \\ &\leq \lambda^{-1} (\tilde{d}(\bar{f}(x), \bar{f}(y)) + \varepsilon). \end{aligned}$$

Since ε is arbitrary, we conclude that $\tilde{d}(\bar{f}(x), \bar{f}(y)) \geq \lambda \tilde{d}(x, y)$.

Since $\bar{f}(y_0) = y_0$, by the above result $\bar{f}(\tilde{B}_t(y_0)) \supset \tilde{B}_{\lambda t}(y_0)$ for $t \geq 0$.

Let K be as above and let $N_\delta(K) = \{y \in \bar{M} : \tilde{d}(y, K) \leq \delta\}$. As we saw before $N_\delta(K)$ is compact and hence

$$A = \max\{\tilde{d}(\bar{f}(x), \bar{f}(y)) : x, y \in N_\delta(K)\}$$

exists. If $\tilde{d}(x, y) \leq \delta$, then there is $\gamma \in \pi_1(M)$ such that $\gamma(x), \gamma(y) \in N_\delta(K)$, and so

$$\begin{aligned} A &\geq \tilde{d}(\bar{f} \circ \gamma(x), \bar{f} \circ \gamma(y)) \\ &= \tilde{d}(\gamma' \circ \bar{f}(x), \gamma' \circ \bar{f}(y)) \quad (\text{for some } \gamma' \in \pi_1(M)) \\ &= \tilde{d}(\bar{f}(x), \bar{f}(y)) \quad (\text{by (5)}). \end{aligned}$$

Therefore $\tilde{d}(\bar{f}(x), \bar{f}(y)) \geq A$ implies $\tilde{d}(x, y) \geq \delta$. From this fact we see that $M_A(\bar{f}(D)) \leq M_\delta(D)$ for any compact subset D of $\bar{M}$.

Let $x \in \bar{M}$. Then there is $\gamma \in \pi_1(M)$ such that $\gamma\tilde{B}_A(x) \subset N_A(K)$. Since $M_{\delta/3}(\gamma\tilde{B}_A(x)) = M_{\delta/3}(\tilde{B}_A(x))$ by (5), it follows that $M_{\delta/3}(\tilde{B}_A(x)) \leq c'$ where $c' = M_{\delta/3}(N_A(K))$. Let D be a compact set of $\bar{M}$ and choose $E_A \in \mathcal{S}(D, A)$ such that $\#E_A = M_A(D)$. Then $D \subset \bigcup_{x \in E_A} \tilde{B}_A(x)$. Hence we have

$$\begin{aligned} M_\delta(D) &\leq M_\delta\left(\bigcup_{x \in E_A} \tilde{B}_A(x)\right) \\ &\leq \#E_A \max\{M_{\delta/3}(\tilde{B}_A(x)) : x \in E_A\} \\ &\leq M_A(D)c'. \end{aligned}$$

We now calculate that

$$\begin{aligned} M_\delta(\tilde{B}_{\lambda t}(y_0)) &\leq M_\delta(\bar{f}(\tilde{B}_t(y_0))) \leq c' M_A(\bar{f}(\tilde{B}_t(y_0))) \\ &\leq c' M_\delta(\tilde{B}_t(y_0)) \end{aligned}$$

and letting $n = [\log_\lambda t]$

$$\begin{aligned} M_\delta(\tilde{B}_{ct}(y_0)) &\leq c' M_\delta(\tilde{B}_{ct/\lambda}(y_0)) \leq (c')^n M_\delta(\tilde{B}_{ct/\lambda^n}(y_0)) \\ &\leq N(c')^n \quad (\text{where } N = M_\delta(\tilde{B}_{c\lambda}(y_0))) \\ &\leq N(c')^{\log_\lambda t} = Nt^{\log_\lambda c'}. \end{aligned}$$

Therefore $\#B(t) \leq Nt^{\log_\lambda c'}$ by (7).

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