SMALL SPACES WHICH "GENERATE" LARGE SPACES

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ABSTRACT. Let $\underline{S}$ denote the class of Tychonoff spaces. For $X \in \underline{S}$ and $\underline{C} \subseteq \underline{S}$, we say that X generates large $\underline{C}$ -spaces if for every cardinal α there is $Y \in \underline{S}$ such that $Y \in \underline{C}$, $X \subseteq Y$, and every $Z \in \underline{C}$ with $X \subseteq Z \subseteq Y$ satisfies $|Z| > \alpha$. For classes $\underline{C}$ which satisfy certain mild and natural conditions, we show for each $X \in \underline{S}$ that X generates large $\underline{C}$ -spaces iff there is no weakly free $\underline{C}$ -space over X —i.e., no space Y such that $X \subseteq Y \in \underline{C}$ and every continuous $f: X \rightarrow Z \in \underline{C}$ extends to a continuous function $\bar{f}: Y \rightarrow Z$. Among the classes $\underline{C} \subseteq \underline{S}$ which satisfy these conditions for every $X \notin \underline{C}$ are the class of pseudocompact Tychonoff spaces and the class of almost compact (= absolutely C^* -embedded) Tychonoff spaces.

1. Introduction. For convenience and ease of exposition, we restrict our attention throughout this paper to Tychonoff spaces, i.e., to completely regular, Hausdorff spaces. We denote by $\underline{S}$ the class of all such spaces.

It is a familiar consequence of the adjoint functor theorem of Freyd [5, 6] that for certain subclasses $\underline{C}$ of $\underline{S}$ there is for every $X \in \underline{S}$ a free $\underline{C}$ -object over X —that is, a space $Y \in \underline{S}$ such that $Y \in \underline{C}$ and X is a dense subspace of Y and every continuous $f: X \rightarrow Z$ with $Z \in \underline{C}$ extends uniquely to a continuous function $\bar{f}: Y \rightarrow Z$. It is shown by van der Slot [14] that every $X \in \underline{S}$ generates a free $\underline{C}$ -space if and only if $\underline{C}$ is a *universal topological property* in the sense that (i) every compact space has $\underline{C}$; (ii) the product of every set of spaces in $\underline{C}$ is again in $\underline{C}$; and (iii) every closed subspace of a space in $\underline{C}$ is in $\underline{C}$. Within our context (Tychonoff spaces), the universal topological properties of van der Slot are exactly the *topological extension properties* defined and studied by Woods [15]. Other relevant references are [13, 14, 10, 11, 4, and 2].

Given $X \in \underline{S}$ and a universal topological property $\underline{C} \subseteq \underline{S}$, the definition of the free $\underline{C}$ -object Y over X is essentially a three-step procedure: (1) One finds a set $\{\langle f_i, Z_i \rangle: i \in I\}$ of ordered pairs with (in each case) $Z_i \in \underline{C}$ and f_i continuous from X into Z_i , with this set being chosen sufficiently large to ensure that for every continuous $f: X \rightarrow Z \in \underline{C}$ there is $i \in I$ so that the pairs $\langle f, \text{cl}_Z f[X] \rangle$ and $\langle f_i, Z_i \rangle$ are naturally isomorphic; (2) one defines

$$e: X \rightarrow P = \prod_{i \in I} Z_i$$

by $(e(x))_i = f_i(x)$; and (3) identifying X with its homeomorph $e[X]$, one writes $Y = \text{cl}_P X$. In the present paper, which originates with a question concerning a

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class $\underline{C}$ that is not a universal topological property (see §6 for some specifics), no set of pairs as in (1) above is available. For certain spaces X there will be no free $\underline{C}$ -space over X , and we are led to consider the following less demanding condition.

1.1. DEFINITION. Let $\underline{C} \subseteq \underline{S}$ and $X \in \underline{S}$. A space Y is a *weakly free $\underline{C}$ -space over X* provided (i) $Y \in \underline{C}$; (ii) $X \subseteq Y$; and (iii) for every continuous $f: X \rightarrow Z$ with $Z \in \underline{C}$ there is a continuous function $\bar{f}: Y \rightarrow Z$ such that $f \subseteq \bar{f}$.

The condition just defined is weaker than the condition that Y be a free $\underline{C}$ -space over X in that the extension $\bar{f}$ of f is not assumed unique. In the following section we develop a criterion for the existence of weak $\underline{C}$ -spaces.

2. On the "generation" of large spaces. If Y is a Hausdorff space in particular, if $Y \in \underline{S}$ —then for every $A \subseteq Y$ one has $|\text{cl}_Y A| \leq 2^{2^{|A|}}$; thus if $\underline{C} \subseteq \underline{S}$ and every closed subspace of a space in $\underline{C}$ is in $\underline{C}$ then for $A \subseteq Y \in \underline{C}$ there is $Z \in \underline{C}$ such that $A \subseteq Z \subseteq Y$ and $|Z| \leq 2^{2^{|A|}}$. This phenomenon suggests the following definition.

2.1. DEFINITION. Let $\underline{C} \subseteq \underline{S}$ and $X \in \underline{S}$. Then X *generates large $\underline{C}$ -spaces* if for every cardinal number α there is $Y \in \underline{S}$ such that (i) $Y \in \underline{C}$; (ii) $X \subseteq Y$; and (iii) if $X \subseteq Z \subseteq Y$ and $Z \in \underline{C}$, then $|Z| > \alpha$.

2.2. THEOREM. Let $\underline{C} \subseteq \underline{S}$ and $X \in \underline{S}$ and suppose that $\underline{C}$ is preserved by continuous functions. If X generates large $\underline{C}$ -spaces then there is no weakly free $\underline{C}$ -space over X .

PROOF. Suppose instead that F is a weakly free $\underline{C}$ -space over X , set $\alpha = |F|$, and find $Y \in \underline{C}$ such that $X \subseteq Y$ and such that every $Z \in \underline{C}$ with $X \subseteq Z \subseteq Y$ satisfies $|Z| > \alpha$. Since F is weakly free over X the inclusion function $i: X \rightarrow Y$ extends to a continuous function $\bar{i}: F \rightarrow Y$ and with $Z = \bar{i}[F]$ we have

$$Z \in \underline{C}, \quad X \subseteq Z \subseteq Y, \quad \text{and} \quad |Z| \leq \alpha,$$

a contradiction.

We show in 2.4 that for many classes $\underline{C} \subseteq \underline{S}$ and spaces $X \in \underline{C}$, Theorem 2.2 has a satisfactory converse.

We note first that if $\underline{C} \subseteq \underline{S}$ and $X \in \underline{S}$ and X does not generate large $\underline{C}$ -spaces, then there is a cardinal number α such that if $X \subseteq Y \in \underline{C}$ then there is $Z \in \underline{C}$ such that $X \subseteq Z \subseteq Y$ and $|Z| \leq \alpha$; let us call such a cardinal number α a $\underline{C}$ -cap for X .

Modifying terminology suggested by Engelking and Mrówka [3], we say for $\underline{C} \subseteq \underline{S}$ and $X \in \underline{S}$ that X is $\underline{C}$ -regular if there is $Y \in \underline{C}$ such that $X \subseteq Y$.

The following lemma is now available.

2.3. LEMMA. Let $\underline{C} \subseteq \underline{S}$ and let $\underline{C}$ be preserved by continuous functions and by finite products. Let $X \in \underline{S}$ be $\underline{C}$ -regular, suppose that X does not generate large $\underline{C}$ -spaces, and let $f: X \rightarrow X' \in \underline{S}$ be a continuous surjection. Then

- (a) X' does not generate large $\underline{C}$ -spaces and
- (b) any $\underline{C}$ -cap for X is a $\underline{C}$ -cap for X' .

PROOF. It is enough to show that if $X' \subseteq Y' \in \underline{C}$ and α is a $\underline{C}$ -cap for X , then there is $Z' \in \underline{C}$ such that $X' \subseteq Z' \subseteq Y'$ and $|Z'| \leq \alpha$. For this, let $X \subseteq Y \in \underline{C}$, let Γ denote the graph of f , and note that since Γ is a subspace of $Y \times Y' \in \underline{C}$ and X is homeomorphic to Γ there is $Z \in \underline{C}$ such that $X \subseteq Z \subseteq Y \times Y'$ and $|Z| \leq \alpha$. The space $Z' = \pi_{Y'}[Z]$ is as required.

2.4. THEOREM. *Let $\underline{C} \subseteq \underline{S}$ and let $\underline{C}$ be preserved by continuous functions and by arbitrary products. If $X \in \underline{S}$ is $\underline{C}$ -regular and X does not generate large $\underline{C}$ -spaces, then there is a weakly free $\underline{C}$ -space over X .*

PROOF. (This is a repetition, with minor but necessary modifications, of the three-step procedure outlined above in the Introduction.) Let $\{\langle f_i, Z_i \rangle : i \in I\}$ be a listing of all ordered pairs with $Z_i \in \underline{C}$, f_i a continuous function from X into Z_i , and with (as a set) $Z_i \subseteq \alpha$ where α is a $\underline{C}$ -cap for X . (This last requirement, a technical condition stronger than $|Z_i| \leq \alpha$, assures us that the indexed family in question is indeed a set and not a proper class.) Since X is $\underline{C}$ -regular the function

$$e: X \rightarrow P = \prod_{i \in I} Z_i$$

defined by $(e(x))_i = f_i(x)$ is a homeomorphism into P . Suppressing mention of e , we write $X \subseteq P \in \underline{C}$ and we claim that P is a weakly free $\underline{C}$ -space over X . Indeed if $f: X \rightarrow W$ with f continuous and $W \in \underline{C}$ then according to Lemma 2.3 there is $Z \in \underline{C}$ such that $f[X] \subseteq Z \subseteq W$ and $|Z| \leq \alpha$. Assuming without loss of generality the set-theoretic inclusion $Z \subseteq \alpha$, we find $i \in I$ such that

$$\langle f, Z \rangle = \langle f_i, Z_i \rangle;$$

the function $\bar{f} = \pi_i: P \rightarrow Z_i = Z \subseteq W$ is continuous and satisfies $f \subseteq \bar{f}$, as required.

Theorems 2.2 and 2.4 may be combined as follows.

2.5. THEOREM. *Let $\underline{C} \subseteq \underline{S}$, let $\underline{C}$ be preserved by continuous functions and by arbitrary products, and let $X \in \underline{S}$ be $\underline{C}$ -regular. Then there is a weakly free $\underline{C}$ -space over X if and only if X does not generate large $\underline{C}$ -spaces.*

In the next two sections we apply the foregoing abstract considerations to two specific, familiar classes $\underline{C}$.

3. The class of pseudocompact spaces. For an ordinal number ξ , we denote by $W(\xi)$ the set ξ in its order topology. Each space $W(\xi)$ is locally compact, and $W(\xi)$ is compact if and only if ξ is a successor ordinal; if $W(\xi)$ is not compact then the compact space $W(\xi + 1)$ is (naturally homeomorphic with) the one-point compactification of $W(\xi)$. For each ordinal ξ one has $\xi \notin W(\xi)$ and $\xi \in W(\xi + 1)$.

Of course every infinite cardinal is a limit ordinal. The *cofinality* of an infinite cardinal γ , denoted $\text{cf}(\gamma)$, is the least cardinal α for which there is a set $\{\gamma_\xi: \xi < \alpha\}$ of cardinal numbers such that

$$\gamma_\xi < \gamma \quad \text{for each } \xi < \alpha \quad \text{and} \quad \sum_{\xi} \gamma_\xi = \gamma.$$

As usual (see [12 or 8]), a space X is said to be *pseudocompact* if each continuous, real-valued function on X is bounded. We write

$$\underline{PS} = \{X \in \underline{S}: X \text{ is pseudocompact}\}.$$

The following theorem is undoubtedly well known. A proof may be patterned on the argument given in [8, (5.12)] for the case $\gamma = \omega_1 = \aleph_1$.

3.1. THEOREM. *For each infinite cardinal number γ , the following statements are equivalent.*

- (a) $\text{cf}(\gamma) > \omega$;
- (b) $W(\gamma) \in \underline{PS}$;
- (c) $\beta(W(\gamma)) = W(\gamma + 1)$.

We show now that for $\underline{C} = \underline{PS}$ and $X \notin \underline{C}$, there is no weakly free $\underline{C}$ -space over X .

3.2. THEOREM. *Let $X \in \underline{S}$ and $X \notin \underline{PS}$. Then X generates large $\underline{PS}$ -spaces.*

PROOF. Given an (infinite) cardinal number α , choose a cardinal γ such that $\text{cf}(\gamma) > \alpha$ and define

$$Y = (W(\gamma) \times \beta X) \cup (\{\gamma\} \times X)$$

with the topology inherited from $W(\gamma + 1) \times \beta X$; note that Y may be written in the form

$$Y = (W(\gamma + 1) \times \beta X) \setminus (\{\gamma\} \times (\beta X \setminus X)).$$

From $\text{cf}(\gamma) > \omega$ and 3.1 above follows $W(\gamma) \in \underline{PS}$. Since βX is compact we have $W(\gamma) \times \beta X \in \underline{PS}$ and hence $Y \in \underline{PS}$; further, from [9 and 7] we have

$$\beta Y = \beta(W(\gamma) \times \beta X) = \beta(W(\gamma)) \times \beta X = W(\gamma + 1) \times \beta X.$$

Since X is homeomorphic with the subspace

$$X' = \{\gamma\} \times X$$

of Y , to complete the proof it is enough to show that if Z is a space such that $X' \subseteq Z \subseteq Y$ and $|Z| \leq \alpha$, then $Z \notin \underline{PS}$. Given such Z , write

$$Z' = Z \cap \pi^{-1}(\{\gamma\}) = Z \cap X' = X'$$

with π the projection from βY onto $W(\gamma + 1)$, and note that since

$$|Z \setminus Z'| \leq |Z| < \text{cf}(\gamma) \quad \text{and} \quad \pi[Z \setminus Z'] \subseteq \gamma$$

there is $\xi < \gamma$ such that $\pi[Z \setminus Z'] \subseteq \xi$. It follows that $Z \setminus Z'$ is open-and-closed in Z . Thus Z' , which is the nonpseudocompact space X' , is open-and-closed in Z ; thus $Z \notin \underline{PS}$.

3.3. COROLLARY. *If the space X is not pseudocompact, then no pseudocompact space Y containing X has the property that every continuous function from X to a pseudocompact space Z extends continuously from Y to Z .*

PROOF. The class $\underline{PS}$ is preserved by continuous functions, so 2.2 applies.

4. The class of almost-compact spaces. Following [8], we say that a space X is *almost compact* if $|\beta X \setminus X| \leq 1$. We write

$$\underline{ACS} = \{X \in \underline{S} : X \text{ is almost compact}\}.$$

It is easy to see (as in [8]) that for each $X \in \underline{S}$, the following conditions are equivalent:

- (a) $X \in \underline{ACS}$;
- (b) for each $Y \in \underline{S}$, if $X \subseteq Y$ then X is C^* -embedded in Y ;
- (c) for each $Y \in \underline{S}$, if $X \subseteq Y$ then $\beta X = \text{cl}_{\beta Y} X$.

In view of the equivalence (a) $\Leftrightarrow$ (b), almost compact spaces are often called *absolutely C^* -embedded* spaces.

4.1. LEMMA. Let $X \in \underline{S}$ and $X \notin \underline{ACS}$. There are a compactification BX of X and $r \in BX \setminus X$ such that X is C^* -embedded in the space $F = BX \setminus \{r\}$ and $F \notin \underline{ACS}$.

PROOF. Choose distinct $p, q \in \beta X \setminus X$, write $F = \beta X \setminus \{p, q\}$, and let $BX = F \cup \{r\}$ be the 1-point compactification of F .

For later use we note that the space F of 4.1 is not C^* -embedded in BX . It follows that no space W satisfying $X \subseteq W \subseteq F$ is C^* -embedded in BX .

We want to show that every space which is not absolutely C^* -embedded generates large absolutely C^* -embedded spaces. The following theorem shows a bit more.

4.2. THEOREM. If the space X is not absolutely C^* -embedded, then for every cardinal α there is $Y \in \underline{S}$ such that

- (a) $X \subseteq Y$, and
- (b) if $X \subseteq Z \subseteq Y$ and Z is C^* -embedded in Y , then $|Z| > \alpha$.

Further, one may arrange $Y \in \underline{ACS}$.

PROOF. Choose BX , F and r as in 4.1, choose a cardinal γ such that $\text{cf}(\gamma) > \alpha$, and define

$$Y = (W(\gamma) \times BX) \cup (\{\gamma\} \times F)$$

with the topology inherited from $W(\gamma+1) \times BX$; note that Y may be written in the form

$$Y = (W(\gamma+1) \times BX) \setminus \{(\gamma, r)\}.$$

As in the proof of 3.2 we have $W(\gamma) \times BX \in \underline{PS}$ and $Y \in \underline{PS}$, and hence

$$\beta Y = W(\gamma+1) \times BX = Y \cup \{(\gamma, r)\};$$

thus $Y \in \underline{ACS}$. Now define

$$X' = \{\gamma\} \times X \quad \text{and} \quad F' = \{\gamma\} \times F$$

and let $X' \subseteq Z \subseteq Y$ with $|Z| \leq \alpha$. We will show that Z is not C^* -embedded in Y . For this, since $\beta Y = W(\gamma+1) \times BX$, it is enough to show that Z is not C^* -embedded in $W(\gamma+1) \times BX$. Write

$$Z' = Z \cap \pi^{-1}(\{\gamma\}) = Z \cap F'$$

with π the projection from βY onto $W(\gamma+1)$ and note that since

$$|Z \setminus Z'| \leq |Z| < \text{cf}(\gamma) \quad \text{and} \quad \pi[Z \setminus Z'] \subseteq \gamma$$

there is $\xi < \gamma$ such that $\pi[Z \setminus Z'] \subseteq \xi$. It follows that $Z \setminus Z'$ and Z' are (complementary) open-and-closed subsets of Z , with $X' \subseteq Z' \subseteq F'$. From this latter condition and the remark after 4.1 it follows that Z' is not C^* -embedded in $\{\gamma\} \times BX$. *A fortiori* Z is not C^* -embedded in $W(\gamma+1) \times BX$.

4.3. COROLLARY. Let $X \in \underline{S}$ and $X \notin \underline{ACS}$. Then X generates large ACS-spaces.

PROOF. Given a cardinal number α , find Y as in 4.2. It is clear from (b) that no $Z \in \underline{ACS}$ with $X \subseteq Z \subseteq Y$ can satisfy $|Z| \leq \alpha$.

4.4. COROLLARY. *If the space X is not almost compact, then no almost compact space Y containing X has the property that every continuous function from X to an almost compact space Z extends continuously from Y into Z .*

PROOF. In order to apply 2.2 we must check that if $W \in \underline{ACS}$ and $f: W \rightarrow W' \in \underline{S}$ is a continuous surjection, then $W' \in \underline{ACS}$. The Stone extension

$$\bar{f}: \beta W \rightarrow \beta W'$$

of f satisfies $\beta W' \setminus W' \subseteq \bar{f}[\beta W \setminus W]$, so that

$$|\beta W' \setminus W'| \leq |\beta W \setminus W| \leq 1,$$

as required.

5. A generalization to higher cardinals. The class $\underline{ACS}$ of almost compact spaces (equivalently: absolutely C^* -embedded spaces) is familiar to students of the textbook [8]. We note in this section that Corollary 4.3, asserting that every space $X \notin \underline{ACS}$ generates large $\underline{ACS}$ -spaces, has a full and natural generalization to higher cardinals. Since the classes $\underline{R}(\delta)$ seem a trifle artificial and the proof follows a pattern by now familiar, we content ourselves here with a brief treatment.

5.1. NOTATION. For each cardinal number δ we write

$$\underline{R}(\delta) = \{X \in \underline{S}: \text{each compact } K \subseteq \beta X \setminus X \text{ satisfies } |K| < \delta\}.$$

We note that $\underline{R}(2)$ is exactly the class $\underline{ACS}$.

5.2. THEOREM. *Let δ be a cardinal number such that $\delta \geq 2$, and let $X \in \underline{S}$ and $X \notin \underline{R}(\delta)$. Then*

- (a) X generates large $\underline{R}(\delta)$ -spaces; and
- (b) there is no weakly free $\underline{R}(\delta)$ -space over X .

PROOF. (a) Let compact $K \subseteq \beta X \setminus X$ satisfy $|K| \geq \delta$, define $F = \beta X \setminus K$. let $BX = F \cup \{r\}$ be the 1-point compactification of the locally compact space F and note that every space W such that $X \subseteq W \subseteq F$ satisfies $K \subseteq \beta W \setminus W$ and hence $W \notin \underline{R}(\delta)$.

Now given an infinite cardinal α choose γ so that $\text{cf}(\gamma) > \alpha$, write

$$\begin{aligned} Y &= (W(\gamma) \times BX) \cup (\{\gamma\} \times F) \\ &= (W(\gamma + 1) \times BX) \setminus \{\langle \gamma, r \rangle\}, \end{aligned}$$

and note as before that $Y \in \underline{ACS} \subseteq \underline{R}(\delta)$. With

$$X' = \{\gamma\} \times X \quad \text{and} \quad F' = \{\gamma\} \times F$$

we see as before that every space Z such that $X' \subseteq Z \subseteq Y$ and $|Z| \leq \alpha$ has a relatively open-and-closed subspace Z' such that $X' \subseteq Z' \subseteq F'$. Then from $Z' \notin \underline{R}(\delta)$ follows $Z \notin \underline{R}(\delta)$, as required.

(b) The argument of 4.4 shows that $\underline{R}(\delta)$ is preserved under continuous functions, so (b) follows from (a) and 2.2.

6. Remarks and acknowledgments. Let $\underline{PAG}$ denote the class of pseudo-compact Abelian (Hausdorff) topological groups. Following [1], we say for $X \in \underline{S}$ that G is a *free $\underline{PAG}$ -group over X* provided (i) $G \in \underline{PAG}$; (ii) $X \subseteq G$; and (iii) for every continuous function $f: X \rightarrow H$ with $H \in \underline{PAG}$ there is a unique continuous

homomorphism $\bar{f}: G \rightarrow H$ such that $f \subseteq \bar{f}$. The point of departure of the present paper is [1], where it is shown *inter alia* that for $\emptyset \neq X \in \underline{S}$ there is no free PAG-group over X . Both here and in [1] it is left as an open question whether for $\emptyset \neq X \in \underline{S}$ there is a weakly free PAG-group over X in the sense of the following definition.

6.1. DEFINITION. Let $X \in \underline{S}$. Then

(a) X generates large PAG-groups if for every cardinal number α there is a topological group G such that (i) $G \in \underline{PAG}$; (ii) $X \subseteq G$; and (iii) if $X \subseteq H \subseteq G$ and $H \in \underline{PAG}$, then $|H| > \alpha$.

(b) G is a weakly free PAG-group over X provided (i) $G \in \underline{PAG}$; (ii) $X \subseteq G$; and (iii) for every continuous $f: X \rightarrow H$ with $H \in \underline{PAG}$ there is a continuous homomorphism $\bar{f}: G \rightarrow H$ such that $f \subseteq \bar{f}$.

Essentially as in 2.2 above, one can show that if $X \in \underline{S}$ and X generates large PAG-groups, then there is no weakly free PAG-group over X . The following conjecture is attractive.

6.2. CONJECTURE. Every nonempty space generates large PAG-groups.

6.3. ACKNOWLEDGMENTS. The proof of Theorem 4.2 incorporates a suggestion made by John R. Isbell. I am pleased also to acknowledge helpful conversations with Anthony W. Hager, Paul Johnson, Fred E. J. Linton and Jan van Mill.

6.4. NOTE (ADDED OCTOBER, 1987). I have received a letter from Michael Tkachenko of Balokovo, USSR, announcing that that he has shown the existence, for every infinite cardinal α , of a group $G \in \underline{PAG}$ such that every infinite pseudocompact subset Y of G satisfies $|Y| \geq \alpha$. Since every infinite (Tychonoff) space contains a copy of the countably infinite discrete space ω , it is immediate from Tkachenko's theorem that (in the terminology of 6.1(a)) the space ω and hence every infinite $X \in \underline{S}$ —generates large PAG-groups.

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