CONTINUITY OF ADDITIVE κ -METRIC FUNCTIONS AND METRIZATION OF κ -METRIC SPACES

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ABSTRACT. For an additive κ -metric space X with an $s(x)$ -continuous κ -metric $d(x, C)$, we prove that X is metrizable, and that if $d(x, C)$ is locally regular, then $z(x, y)$ is bicontinuous, and $\rho(x, y) = z(x, y) + z(y, x)$ is a metric on X which agrees with the topology of X .

Introduction. Dranišnikov [2] defined the additivity of a κ -metric space, which is introduced by Ščepin [4,5] as a generalization of metric spaces and locally compact groups, and stated that a compact additive κ -metric space is metrizable. In the previous paper [3], we proved that an additive κ -metric space X is metrizable if X is either pseudocompact or stratifiable. In this paper, investigating the relationship among various kinds of continuities of a κ -metric $d(x, C)$, we prove the following:

MAIN THEOREM. *Let (X, d) be an additive κ -metric space with an $s(x)$ -continuous κ -metric $d(x, C)$. Then we have:*

- (1) X is metrizable.
- (2) *If $d(x, C)$ is locally regular, then $z(x, y)$ is bicontinuous and $\rho(x, y) = z(x, y) + z(y, x)$ is a metric on X which agrees with the topology of X .*

Throughout this paper, we mean by (X, d) an additive κ -metric space.

1. Definition and preliminaries.

1.1. DEFINITION. Let X be a topological space and let $\text{Rc}(X)$ be the collection of regular closed subsets of X . (X, d) is an *additive κ -metric space* [2,4,5] if X is Tychonoff and d is a mapping of $X \times \text{Rc}(X)$ to $[0, 1]$ satisfying the following:

- (1) $d(x, C) = 0 \Leftrightarrow x \in C$.
- (2) $C \subset D \Rightarrow d(x, C) \geq d(x, D)$.
- (3) $d(x, C)$ is continuous with respect to x for each C (abbreviated by *x -continuous*).
- (4) $d(x, \text{cl}(\bigcup \{C_a; a \in A\})) = \inf\{d(x, C_a); a \in A\}$. $d(x, C)$ is called an *additive κ -metric function* (abbreviated by *additive κ -metric*) on X . For a closed set F of (X, d) , we put $\text{Rc}(F, X) = \{C \in \text{Rc}(X); F \subset C\}$, $\text{Rci}(X, F) = \{C \in \text{Rc}(X); F \subset \text{int } C\}$, $\text{Rci}(x) = \text{Rci}(X, \{x\})$, $\text{Ro}(X) = \{U; U \text{ is regular open}\}$, $z(x, F) = \sup\{d(x, C); C \in \text{Rci}(X, F)\}$, and $\mathbb{N}$ = the set of positive integers. We introduce in $\text{Rc}(X)$ the *Vietoris topology* in the following sense: the basic open sets in $\text{Rc}(X)$ are the sets of the form $\langle U_1, U_2, \dots, U_n \rangle$ where $U_i \in \text{Ro}(X)$, $1 \leq i \leq n$, and $\langle U_1, U_2, \dots, U_n \rangle = \{F \in \text{Rc}(X); F \subset \bigcup U_i \text{ and } F \cap U_i \neq \emptyset \text{ for each } i\}$. It

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is easily verified that $\text{Rc}(X)$ is Hausdorff. We say that $d(x, C)$ is *C-continuous* (*bicontinuous*) if $d(x, C)$ is continuous with respect to C (jointly continuous with respect to x and C). $d(x, C)$ is *regular* [4] if it satisfies the following:

(R) $d(x, C) \leq d(x, D) + \sup\{d(s, C); s \in D\}$ for x, C and D . We say that $d(x, C)$ is *locally regular* if for x, y, t and $\varepsilon > 0$, there are $C_0 \in \text{Rci}(y)$ and $D_0 \in \text{Rci}(t)$ such that

(LR) $d(x, C) \leq d(x, D) + \sup\{d(s, C); s \in D\} + \varepsilon$ for $C \in \text{Rci}(y)$ with $C \subset C_0$ and $D \in \text{Rci}(t)$ with $D \subset D_0$.

It is evident that if $d(x, C)$ is regular, it is locally regular. We say that $d(x, C)$ is $s(x)$ ($s(y)$ and $s(x, y)$, resp.) -*continuous* if for $\varepsilon > 0$, x and y there are $U \in \text{Rci}(x)$ and $C \in \text{Rci}(y)$ satisfying the following condition ($(s(x))$, $(s(y))$ and $(s(x, y))$, resp.).

$(s(x))$ $|d(x, D) - d(x', D)| < \varepsilon$ for $x' \in U$ and $D \in \text{Rci}(y)$ with $D \subset C$.

$(s(y))$ $|d(x, D) - d(x, E)| < \varepsilon$ for $D \in \text{Rci}(y)$ with $D \subset C$ and $E \in \text{Rc}(X)$ with $E \subset C$.

$(s(x, y))$ $|d(x, D) - d(x', E)| < \varepsilon$ for $x' \in U$, $D \in \text{Rci}(y)$ with $D \subset C$ and $E \in \text{Rc}(X)$ with $E \subset C$.

1.2 LEMMA. For (X, d) we have

(1) $z(x, F) = 0 \Leftrightarrow x \in F$, and $z(x, F) = \inf\{z(x, y); y \in F\}$ [3].

(2) $z(x, C) \leq d(x, C)$ for $C \in \text{Rc}(X)$.

(3) $\{U(x, n); n \in \mathbb{N}\}$ is an open base at x where $U(x, n) = \text{int } S(x, n)$ and $S(x, n) = \{t; z(x, t) < 1/n\}$.

(4) For $\varepsilon > 0$, x and y , there are $C \in \text{Rci}(x)$ and $D \in \text{Rci}(y)$ such that $z(x, y) - \varepsilon < z(x', y')$ for $x' \in C$ and $y' \in D$.

PROOF. (4) Let $x \neq y$. By the definition of $z(x, y)$, there is $E \in \text{Rci}(y)$, $x \notin E$ with $z(x, y) - \varepsilon/2 < d(x, E)$. Since $d(x, E)$ is x -continuous, there is $C \in \text{Rci}(x)$ such that $|d(x, E) - d(x', E)| < \varepsilon/2$ for $x' \in C$. If $y' \in \text{int } E$, $d(x', E) \leq z(x', y')$ and hence $z(x, y) - \varepsilon < z(x', y')$ for $x' \in C$ and $y' \in D \in \text{Rci}(y)$ with $D \subset \text{int } E$.

2. Example. In this section, we show that the Sorgenfrey line $\mathbf{R}$ is a non-metrizable additive κ -metric space such that $d(x, C)$ is regular and bicontinuous, but not $s(x)$ -continuous. For $x \in \mathbf{R}$ and $C \in \text{Rc}(\mathbf{R})$, put $C(x^+) = C \cap [x, \infty)$ and $C^+(x) = \inf\{t \in C(x^+)\}$ ($= a$) if $C(x^+) \neq \emptyset$. We define $d(x, C) = \min\{a - x, 1\}$ if $C(x^+) \neq \emptyset$, and $d(x, C) = 1$ otherwise. In [6] it is proved that $(\mathbf{R}, d)$ is an additive κ -metric space, and $d(x, C)$ is regular.

(1) $d(x, C)$ is bicontinuous. Let $x \in \mathbf{R}$ and $C \in \text{Rc}(\mathbf{R})$.

Case 1. $C(x^+) = \emptyset$. Then $d(x, C) = 1$. Put $U = (-\infty, x)$ and $V = [x, x + 1)$. Then $d(x', D) = 1$ for $x' \in V$ and $D \in \langle U \rangle$, and hence $d(x, C) = d(x', D)$.

Case 2. $C(x^+) \ni x$. Then $d(x, C) = 0$. Since $\text{cl}(\text{int } C) \ni x$ for a given $\varepsilon > 0$, $A = [x, x + \varepsilon) \cap C \neq \emptyset$. Take $\delta > 0$ and $y \in A$ such that $y - x > \delta > 0$ and $[y - \delta, y + \delta] \subset (x, x + \varepsilon)$. Put $U_1 = (-\infty, y - \delta)$, $U_2 = [y - \delta, y + \delta)$, $U_3 = [y + \delta, \infty)$, and $U = [x, y - \delta)$. For $D \in \langle U_1, U_2, U_3 \rangle$ and $t \in U$. We have $d(t, D) \leq d(t, D \cap U_2) < \varepsilon$, and hence $|d(t, D) - d(x, C)| < \varepsilon$.

Case 3. $C(x^+) \neq \emptyset$ and $C(x^+) \not\ni x$. Let $0 < 2\varepsilon < \min\{1, a - x\}$. Put $U_1 = (-\infty, x)$, $U_2 = [a - \varepsilon, a + \varepsilon)$, $U_3 = [a + \varepsilon, \infty)$ and $U = [x, x + \varepsilon)$. For $t \in U$ and $D \in \langle U_1, U_2, U_3 \rangle$, $D \cap U_2 \neq \emptyset$, and $a - x - \varepsilon < d(t, D) \leq d(t, U_2 \cap D) \leq \min\{1, (a + \varepsilon) - x\}$, and hence $|d(t, D) - d(x, C)| < \varepsilon$.

(2) $d(x, C)$ is not $s(x)$ -continuous. Let $x = y$. For any open set $U \ni x$, take $D \in \text{Rci}(y)$ with $D \subset C$ and $x' \in U - D$. Then $d(x', D) = 1$ and $d(x, D) = 0$, and hence the condition $(s(x))$ in 1.1 does not hold.

3. Local regularity and C -continuity of $d(x, C)$.

3.1. THEOREM. For (X, d) , the following are equivalent.

- (1) $d(x, C)$ is C -continuous.
- (2) $z(x, C) = d(x, C)$ for $C \in \text{Rc}(X)$.
- (3) $z(x, y) = \sup\{d(x, C); y \in C \in \text{Rc}(X)\}$.
- (4) $d(x, C)$ is $s(y)$ -continuous.
- (5) $z(x, y)$ is y -continuous.

PROOF. (1) $\Rightarrow$ (2). For $\varepsilon > 0$ and $C \in \text{Rc}(X)$, there is an open set $\langle U_1, U_2, \dots, U_n \rangle \ni C$ such that $|d(x, C) - d(x, D)| < \varepsilon$ for $D \in \langle U_1, U_2, \dots, U_n \rangle$. Since X is a κ -metric space, we can take $D \in \text{Rci}(X, C) \cap \langle U_1, U_2, \dots, U_n \rangle$. But $d(x, C) < d(x, D) + \varepsilon$ implies $d(x, C) < z(x, C) + \varepsilon$ by the definition of $z(x, C)$, so $d(x, C) \leq z(x, C)$. By 1.2(2), we have $d(x, C) = z(x, C)$.

(2) $\Rightarrow$ (3). $\sup\{d(x, C); y \in C \in \text{Rc}(X)\} = \sup\{\sup\{d(x, D); D \in \text{Rci}(X, C)\}; C \in \text{Rc}(X)\} = \sup\{d(x, C); C \in \text{Rci}(y)\} = z(x, y)$.

(3) $\Rightarrow$ (4). Let $\{C_n \in \text{Rci}(y); n \in \mathbb{N}\}$ be a base at y with $C_{n+1} \subset C_n$, and $a = z(x, y)$. Then $a = \lim d(x, C_n)$, and hence we can assume that $a - \varepsilon/2 < d(x, C_n) \leq a$ for $n \in \mathbb{N}$. Suppose that (4) does not hold. Then for each n , $|d(x, D_n) - d(x, E_n)| \geq \varepsilon$ for some $D_n \in \text{Rci}(y)$ with $D_n \subset C_n$ and some $E_n \subset C_n$. Since $d(x, C_n) \leq d(x, D_n) \leq a$ and $d(x, C_n) \leq d(x, E_n)$, we have $d(x, E_n) > a + \varepsilon/2$. On the other hand, $y \in \text{cl}(\bigcup E_n) (= A)$ and $d(x, A) = \inf\{d(x, E_n); n \in \mathbb{N}\} \geq a + \varepsilon/2$, but $a = z(x, y) \geq d(x, A)$ by (3), a contradiction.

(4) $\Rightarrow$ (5). Applying $E \in \text{Rci}(y')$ to E in $(s(y))$ of 1.1 where $y' \in \text{int } C$, $|z(x, y) - z(x, y')| \leq \varepsilon$, and hence $z(x, y)$ is y -continuous.

(5) $\Rightarrow$ (2). Let $C \in \text{Rc}(X)$ and $x \in X$. By 1.2(1,2), $d(x, C) \geq z(x, C) = \inf\{z(x, t); t \in C\}$. Since $z(x, y)$ is y -continuous, we have $z(x, C) = \inf\{z(x, t); t \in \text{int } C\}$. But $C \in \text{Rci}(t)$ implies $z(x, t) \geq d(x, C)$, and hence $z(x, C) \leq d(x, C)$, so $d(x, C) = z(x, C)$.

(5) $\Rightarrow$ (1). Let $x \in X$, $\varepsilon > 0$ and $a = d(x, C)$. By (5) $\Rightarrow$ (2), and (5), we have $d(x, C) = z(x, C) = \inf\{z(x, y); y \in C\} = \inf\{z(x, y); y \in \text{int } C\}$, and hence there is $y \in \text{int } C$ with $a \leq z(x, y) < a + \varepsilon$. Let us put $U = \{t; z(x, t) > a - \varepsilon\}$ and $W = \{t; a - \varepsilon < z(x, t) < a + \varepsilon\}$. Then $U \supset W \cup C$. Take $S \in \text{Rci}(y)$ with $S \subset U_1 = W \cap \text{int } C$ and put $U_2 = W \cap (X - S)$. Then $C \in \langle U_1, U_2 \rangle$. Let $D \in \langle U_1, U_2 \rangle$. Put $D_i = D \cap \text{cl } U_i$, $i = 1, 2$. Then $D = D_1 \cup D_2$, and $D, D_i \in \text{Rc}(X)$, $i = 1, 2$. For $t \in U_1 \cap D$, $a - \varepsilon < z(x, t) < a + \varepsilon$, and for $t \in U_2 \cap D$, $a - \varepsilon < z(x, t)$. Since $z(x, D) = d(x, D) = \min\{d(x, D_i); i = 1, 2\} = \min\{\inf\{z(x, t); t \in U_i \cap D\}; i = 1, 2\}$, we have $a - \varepsilon < d(x, D) < a + \varepsilon$, i.e., $|d(x, D) - d(x, C)| < \varepsilon$, and hence $d(x, C)$ is C -continuous.

3.2. THEOREM. For (X, d) , we have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

- (1) $d(x, C)$ is locally regular.
- (2) $z(x, y) \leq z(x, t) + z(t, y)$ for x, y and t .
- (3) $z(x, y)$ is y -continuous.

PROOF. (1) $\Rightarrow$ (2). Suppose that (LR) in 1.1 holds. Let $x, y, t \in X$, D_0, C_0 be as in (LR) and $C \subset C_0$. $d(x, C)$ being x -continuous, there is $D \in \text{Rci}(t)$ such that $|d(t, C) - d(s, C)| < \varepsilon$ for $s \in D$. Thus, $\sup\{d(s, C); s \in D\} \leq d(t, C) + \varepsilon$. Since (LR) holds, we have $d(x, C) \leq d(x, D) + d(t, C) + 2\varepsilon \leq z(x, t) + z(t, y) + 2\varepsilon$, so $z(x, y) \leq z(x, t) + z(t, y) + 2\varepsilon$. Since ε is arbitrary, we have $z(x, y) \leq z(x, t) + z(t, y)$.

(2) $\Rightarrow$ (1). For some $D_0 \in \text{Rci}(t)$ and some $C_0 \in \text{Rci}(y)$, $z(x, t) < d(x, D) + \varepsilon/3$ for $D \in \text{Rci}(t)$ with $D \subset D_0$, and $z(t, y) < d(t, C) + \varepsilon/3$ for $C \in \text{Rci}(y)$ with $C \subset C_0$. Since $d(x, C)$ is x -continuous, there is $D_C \in \text{Rci}(t)$ with $D_C \subset D_0$ such that $d(t, C) < d(s, C) + \varepsilon/3$ for $s \in D \in \text{Rci}(t)$ with $D \subset D_C$, and hence, $d(t, C) < \sup\{d(s, C); s \in D\} + \varepsilon/3$. Thus $d(x, C) \leq z(x, y) \leq z(x, t) + z(t, y) \leq d(x, D) + \varepsilon/3 + d(t, C) + \varepsilon/3 < d(x, D) + \sup\{d(s, C); s \in D\} + \varepsilon$.

(2) $\Rightarrow$ (3). For a given $\varepsilon > 0$, we take $C \in \text{Rci}(y)$ such that $z(x, y) - \varepsilon < z(x, y')$ by 1.2(4), and $z(y, y') < \varepsilon$ for $y' \in C$ by 1.2(3). On the other hand, $z(x, y') \leq z(x, y) + z(y, y') < z(x, y) + \varepsilon$, so $|z(x, y) - z(x, y')| < \varepsilon$.

4. Bicontinuity and continuity of $z(x, y)$.

4.1. THEOREM. For (X, d) , $d(x, C)$ is $s(x)$ -continuous iff $z(x, y)$ is x -continuous.

PROOF. ($\Rightarrow$) Let $\{C_n \in \text{Rci}(y); n \in \mathbb{N}\}$ be a base at y with $C_{n+1} \subset C_n$. Since $z(x, y) = \lim d(x, C_n)$, $|d(x, D) - d(x', D)| < \varepsilon$ implies $|z(x', y) - z(x, y)| \leq \varepsilon$ by the condition $(s(x))$ in 1.1, and hence $z(x, y)$ is x -continuous.

($\Rightarrow$) Let $a = z(x, y)$. Since $z(x, y)$ is x -continuous, for $\varepsilon > 0$, there is $U_1 \in \text{Rci}(x)$ such that $a - \varepsilon < z(x', y) < a + \varepsilon$ for $x' \in U_1$. On the other hand, $z(x, y) = \sup\{d(x, C); C \in \text{Rci}(y)\}$ implies that $a - \varepsilon < d(x, C) \leq a$ for some $C \in \text{Rci}(y)$. Since $d(x, C)$ is x -continuous, $d(x, C) - \varepsilon < d(x', C) < d(x, C) + \varepsilon$ for x' in some $U \in \text{Rci}(x)$ with $U \subset U_1$. Thus we have $a - 2\varepsilon < d(x', C) < a + \varepsilon$, and $d(x', C) \leq d(x', D) \leq z(x', y) < a + \varepsilon$ for $x' \in U$ and $C, D \in \text{Rci}(y)$ with $D \subset C$. Since $a - \varepsilon < d(x, C) \leq d(x, D) \leq a$ for $D \subset C$, and $a - 2\varepsilon < d(x', D) < a + \varepsilon$, we have $|d(x, D) - d(x', D)| < 2\varepsilon$, so $d(x, C)$ is $s(x)$ -continuous.

4.2. THEOREM. For (X, d) , we have (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

(1) $d(x, C)$ is locally regular and $s(x)$ -continuous.

(2) $z(x, y)$ is bicontinuous.

(3) $d(x, C)$ is $s(x, y)$ -continuous.

PROOF. (1) $\Rightarrow$ (2). $d(x, C)$ being locally regular, by 3.2, $z(x', y') \leq z(x', y) + z(y, y')$, and $z(x, y)$ is x -continuous. Hence, for $\varepsilon > 0$, there is an open set $U \ni x$ such that $|z(x, y) - z(x', y)| < \varepsilon/2$ for $x' \in U$. Thus $z(x', y') < z(x, y) + \varepsilon$, so $|z(x', y') - z(x, y)| < \varepsilon$ by 1.2(4). Hence $z(x, y)$ is bicontinuous.

(2) $\Rightarrow$ (3). Let $\varepsilon > 0$ and $a = z(x, y)$. Since $z(x, y)$ is bicontinuous, there are $U_1 \in \text{Rci}(x)$ and $C_1 \in \text{Rci}(y)$ such that $|a - z(x', y')| < \varepsilon/2$ for $x' \in U_1$ and $y' \in C_1$. By the definition of $z(x, y)$, $a - \varepsilon/2 < d(x, C) \leq a$ for some $C \in \text{Rci}(y)$ with $C \subset C_1$. Since $d(x, C)$ is x -continuous, $|d(x', C) - d(x, C)| < \varepsilon/2$ for $x' \in U$ for some $U \in \text{Rci}(x)$ with $U \subset U_1$. Let $D \in \text{Rci}(y)$ with $D \subset C$ and $E \in \text{Rc}(X)$ with $E \subset C$. Take $y' \in \text{int } E$. Then we have $d(x, C) \leq d(x, D) \leq a$ and $d(x', C) \leq d(x', E) \leq z(x', y')$. If $d(x, D) \geq d(x', E)$, $d(x, D) - d(x', E) \leq a - d(x', C) < \varepsilon$. If $d(x', E) > d(x, D)$, $d(x', E) - d(x, D) \leq z(x', y') - d(x, C) < \varepsilon$. Thus we have $|d(x, D) - d(x', E)| < \varepsilon$.

(3) $\Rightarrow$ (2). For $y' \in \text{int } C$, apply $E \in \text{Rci}(y')$ with $E \subset C$ to E in $(s(x, y))$ of 1.1.

5. Proof of the main theorem.

PROOF OF (1). Let F be a closed subset of X . Since $d(x, C)$ is x -continuous and $z(x, F) = \sup\{d(x, C); C \in \text{Rci}(X, F)\}$, $z(x, F)$ is lower semi-continuous. On the other hand, since $z(x, y)$ is x -continuous by 4.1 and $z(x, F) = \inf\{z(x, y); y \in F\}$, $z(x, F)$ is upper semi-continuous. Thus $z(x, F)$ is x -continuous. Moreover, it is easily verified that $z(x, F) = 0 \Leftrightarrow x \in F$ for every closed set F : i.e., $z(x, F)$ is an annihilator of F , and that if E and F are closed, and $E \subset F$, then $z(x, E) \geq z(x, F)$ for every x ; i.e., $z(x, F)$ is monotone. Thus $z(x, F)$ is a continuous monotone annihilator of the family of all closed sets in the sense of Dranišnikov [2], and hence X is stratifiable [1, 2], so X is metrizable [3].

PROOF OF (2). By 4.2, $z(x, y)$ is bicontinuous. The second part of (2) follows then directly.

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