A PRODUCT EXPANSION IN p -ADIC AND OTHER NON-ARCHIMEDEAN FIELDS

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(Communicated by Larry J. Goldstein)

ABSTRACT. An algorithm is introduced and shown to lead to a unique infinite product representation for a given p -adic integer A with leading coefficient 1 as a product

$$A = \prod_{n=1}^{\infty} (1 + b_n p^{r_n})$$

where $1 \leq b_n \leq p-1$, $r_n \in \mathbf{N}$ and $r_{n+1} > r_n$. The degree of approximation by the natural number $(1 + b_1 p^{r_1}) \cdots (1 + b_n p^{r_n})$ is also considered. In addition we derive similar representations for elements of arbitrary complete non-Archimedean fields with discrete valuations.

1. Introduction. Let $\mathbf{Q}$ be the field of rational numbers, p a prime number, and $\mathbf{Q}_p$ the completion of $\mathbf{Q}$ with respect to the p -adic valuation $|\cdot|_p$ defined on $\mathbf{Q}$ by

$$(1.1) \quad |A|_p = \begin{cases} p^{-\alpha} & \text{if } A = p^\alpha(r/s) \text{ where } p \nmid r, p \nmid s, \\ 0 & \text{if } A = 0. \end{cases}$$

$\mathbf{Q}_p$ is the field of p -adic numbers with p -adic valuation $|\cdot|_p$, the extension of the original valuation on $\mathbf{Q}$, which has the properties

$$(1.2) \quad \begin{aligned} |A|_p &\geq 0, \quad |A|_p = 0 \quad \text{iff } A = 0, \\ |AB|_p &= |A|_p \cdot |B|_p, \\ |A + B|_p &\leq \max(|A|_p, |B|_p) \end{aligned}$$

with equality when $|A|_p \neq |B|_p$.

It is familiar that the above non-Archimedean valuation leads to an ultrametric distance function ρ on $\mathbf{Q}_p$, with $\rho(A, B) = |A - B|_p$, making $\mathbf{Q}_p$ into a complete metric space with respect to ρ . As is common, we define the *order* $\nu(A)$ of A by $|A|_p = p^{-\nu(A)}$, and set $\nu(0) = +\infty$. Then

$$(1.3) \quad \begin{aligned} \nu(AB) &= \nu(A) + \nu(B), \quad \nu(A/B) = \nu(A) - \nu(B) \quad \text{if } B \neq 0, \\ \nu(A + B) &\geq \min(\nu(A), \nu(B)) \end{aligned}$$

with equality when $\nu(A) \neq \nu(B)$. Our main attention below will be confined to those p -adic numbers lying in the open unit ball $\mathcal{P}_1 = 1 + p\mathbf{Z}_p$ with centre 1, i.e. the p -adic integers which have p -adic expansions of type $1 + \sum_{n=1}^{\infty} c_n p^n$, $0 \leq c_n \leq p-1$.

Our main result is:

Received by the editors May 15, 1987 and, in revised form, January 4, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 10F45, 12B40.

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(1.4) THEOREM. Every $A \in \mathcal{P}_1$ with $A \neq 1$ has a unique convergent product representation (relative to ρ) of the form

$$A = \prod_{n=1}^{\infty} (1 + b_n p^{r_n}),$$

where $1 \leq b_n \leq p-1$, $r_n \in \mathbf{N}$ and $r_{n+1} > r_n$. Further $\nu(A - P_n) \geq r_{n+1} > n$ where $P_n = (1 + b_1 p^{r_1}) \cdots (1 + b_n p^{r_n}) \in \mathbf{N}$.

This theorem is quite analogous to one on p -adic continued fractions of the form.

$$A = \frac{p^{r_1}}{b_1} + \frac{p^{r_2}}{b_3} + \frac{p^{r_3}}{b_2} + \cdots,$$

which were originally proposed by Mahler [3], and studied further by Schneider [5] and Bundschuh [1].

In the final section, a few remarks are made about extensions to a general complete non-Archimedean field with a discrete valuation.

2. Convergence and approximation. If $1 \neq A \in \mathcal{P}_1$, let $\hat{A}_1 = A$. Then we can write

$$\hat{A}_1 = 1 + b_1 p^{r_1} A'_1,$$

where $r_1 \in \mathbf{N}$, $1 \leq b_1 \leq p-1$ and $A'_1 \in \mathcal{P}_1$. If $\hat{A}_n = 1 + b_n p^{r_n} A'_n$ has already been defined with $1 \neq A'_n \in \mathcal{P}_1$, $1 \leq b_n \leq p-1$, then define

$$\begin{aligned} \hat{A}_{n+1} &= (1 + b_n p^{r_n})^{-1} \hat{A}_n = (1 - b_n p^{r_n} + b_n^2 p^{2r_n} - \cdots)(1 + b_n p^{r_n} + \cdots) \\ &= 1 + b_{n+1} p^{r_{n+1}} A'_{n+1}, \end{aligned}$$

where $A'_{n+1} \in \mathcal{P}_1$, $r_{n+1} > r_n$ and $1 \leq b_{n+1} \leq p-1$.

If $A'_n = 1$, let $A'_{n+1} = 1$ and stop the algorithm.

Then

$$A = \hat{A}_1 = (1 + b_1 p^{r_1}) \hat{A}_2 = \cdots = \hat{A}_{n+1} \prod_{i=1}^n (1 + b_i p^{r_i}).$$

If the procedure does not terminate with some $\hat{A}_{n+1} = 1$, then

$$\nu(\hat{A}_{n+1} - 1) = r_{n+1} \geq 1 + r_n \geq \cdots \geq n + 1 \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Thus $\lim_{n \rightarrow \infty} \hat{A}_{n+1} = 1$, and then $A = \prod_{i=1}^{\infty} (1 + b_i p^{r_i})$, relative to the metric ρ on $\mathbf{Q}_p$. If $P_n = (1 + b_1 p^{r_1}) \cdots (1 + b_n p^{r_n})$, we also have

$$A = (1 + b_{n+1} p^{r_{n+1}} A'_{n+1}) P_n,$$

and so $\nu(A - P_n) = \nu(b_{n+1} p^{r_{n+1}} A'_{n+1} P_n) = r_{n+1} > n$.

3. Uniqueness of representation.

(3.1) LEMMA. Any product $\prod_{n \geq 1} (1 + d_n p^{s_n})$ with $1 \leq d_n \leq p-1$, $s_n \in \mathbf{N}$ and $s_{n+1} > s_n$, converges relative to ρ to an element $B \neq 1$ in $\mathcal{P}_1$ with $B = 1 + d_1 p^{s_1} B'$, $B' \in \mathcal{P}_1$.

PROOF. If $Q_N = \prod_{n=1}^N (1 + d_n p^{s_n})$, then

$$Q_{N+K} - Q_N = Q_N R_{N,K}, \text{ where}$$

$$R_{N,K} = \prod_{n=N+1}^{N+K} (1 + d_n p^{s_n}) - 1 = d_{N+1} p^{s_{N+1}} R'_{N,K}$$

for $R'_{N,K} \in \mathcal{P}_1$, since (s_N) is a strictly increasing sequence. Thus

$$\nu(Q_{N+K} - Q_N) = \nu(Q_N) + \nu(R_{N,K}) = s_{N+1} > N \rightarrow \infty$$

as $N \rightarrow \infty$, independently of $K \geq 1$. Therefore, if the product is infinite, then (Q_N) forms a Cauchy sequence and converges relative to ρ to an element B of the complete metric space $\mathbf{Q}_p$. Since the convergence is relative to ρ , $\nu(B - Q_N) > s_1$ for N sufficiently large, and so $B = 1 + d_1 p^{s_1} B'$ for $B' \in \mathcal{P}_1$, because $Q_N = 1 + R_{0,N} = 1 + d_1 p^{s_1} R'_{0,N}$. Now suppose that $A = \prod_{n \geq 1} (1 + b_n p^{r_n}) = \prod_{n \geq 1} (1 + d_n p^{s_n})$, where $1 \leq b_n \leq p-1$, $1 \leq d_n \leq p-1$, $r_n, s_n \in \mathbf{N}$ and $r_{n+1} > r_n, s_{n+1} > s_n$. Then Lemma 3.1 implies that

$$A = 1 + b_1 p^{r_1} A' = 1 + d_1 p^{s_1} B' \quad \text{for } A', B' \in \mathcal{P}_1.$$

Hence $b_1 = d_1$, $r_1 = s_1$, and we obtain $\prod_{n \geq 2} (1 + b_n p^{r_n}) = \prod_{n \geq 2} (1 + d_n p^{s_n})$. In the same way, $b_2 = d_2$ and $r_2 = s_2$, and thus we successively obtain $b_n = d_n, r_n = s_n$ for all n .

REMARKS. It is interesting to note from the general form of the product for A , and from simple examples within $\mathbf{Q}_p$, such as

$$1 + 3p + 2p^2 = (1 + 3p)(1 + 2p^2)(1 + 4p^3) \cdots, \quad \text{if } p = 5 \text{ say,}$$

that “most” natural numbers have *infinite* representations as in Theorem 1.4, the exceptions being those of the general type P_n of Theorem 1.4. For natural numbers of type P_n , the above algorithm then provides an actual *method* for factorizing P_n in the indicated special way.

A study of the relationship between properties of the p -adic product coefficients and properties of A or of its p -adic series coefficients might perhaps prove interesting. As two little comments in this direction, we note:

(i) If $A = 1 + b_1 p^{r_1} + b_2 p^{r_2} + \cdots$, where $r_1 < r_2, 1 \leq b_i \leq p-1$, then $A = (1 + b_1 p^{r_1})(1 + b_2 p^{r_2}) \cdots$, where however the subsequent factors of the product need *not* necessarily conform to this initial pattern.

(ii) For a simple rational number $(1 - p^r)^{-1}$, with $r \in \mathbf{N}$, the coefficients in the product are given explicitly by the Euler-type identity which occurs in the proof of Proposition 4.1 below.

4. Product expansions for rational numbers. We note that there remains the possibly interesting problem of characterizing the types of products in Theorem 1.4 above which correspond to *rational* numbers $A \in \mathbf{Q}$. In the analogous case of the p -adic continued fractions studied by Schneider [5] and Bundschuh [1], rational numbers are characterized by finite or certain periodic continued fractions. However, since finite expansions as in Theorem 1.4 yield natural numbers, a characterization of nonintegral rationals as well as most natural numbers would therefore have to involve *infinite* products, satisfying possibly special conditions on the exponents r_n . As a partial result in this direction we note:

(4.1) PROPOSITION. Let $A \in \mathcal{P}_1$, $A \neq 1$, with unique product representation

$$A = \prod_{n=1}^{\infty} (1 + b_n p^{r_n}).$$

If $b_n = 1$ and $r_{n+1} = 2r_n$, for all n sufficiently large, then $A \in \mathbf{Q}$.

PROOF. This follows easily from the Euler-type identity

$$\prod_{n=1}^{\infty} (1 + y^{2^{n-1}}) = \frac{1}{1-y} \text{ where } y = p^r, r \in \mathbf{N}.$$

To establish this identity, note that the equation $1 + y = (1 - y^2)/(1 - y)$ leads to

$$\prod_{n=1}^N (1 + y^{2^{n-1}}) = \frac{1-y^2}{1-y} \cdots \frac{1-y^{2^N}}{1-y^{2^{N-1}}} = \frac{1-y^{2^N}}{1-y}.$$

The identity then follows, since $\nu(y^k) = kr \rightarrow \infty$ as $k \rightarrow \infty$, i.e. $y^k \rightarrow 0$ relative to ρ .

5. Products in arbitrary discrete-valued non-Archimedean fields. Let $(K, |\cdot|)$ represent a complete non-Archimedean valued field. Thus the norm $|\cdot|$ of K satisfies properties equivalent to those of equation (1.2) for $|\cdot|_p$. We define the function ν , the valuation of K by

$$\nu(A) = \begin{cases} -\log |A| & \text{if } A \in K, A \neq 0, \\ +\infty & \text{if } A = 0. \end{cases}$$

Then ν has the same properties as those in equation (1.3) for the p -adic valuation. As previously, we can define an ultrametric distance function ρ on K , with $\rho(A, B) = |A - B|$. Let $K^\times$ denote the set $K^\times = \{A \in K, A \neq 0\}$ and for a subset $X \subset K$ let $|X| = \{|A| : A \in X\}$. The *value group* of K is the subgroup $|K^\times|$ of the multiplicative group of the positive real numbers. Suppose now K has a *discrete* valuation. Then (see e.g. Schikhof [4]) there is an element $\pi \in K$ such that

$$(5.1) \quad |\pi| = \max |K^\times| \cap (0, 1)$$

and

$$|K^\times| = \{|\pi|^n : n \in \mathbf{Z}\}.$$

The element π of K plays a role similar to that of p in $\mathbf{Q}_p$. Next we define the *residue class field* of K to be the field

$$k = B_0(1)/B_0(1^-) = \{A \in K : |A| \leq 1\} / \{A \in K : |A| < 1\}$$

where $B_0(1)$ and $B_0(1^-)$ denote the closed and open unit discs with centre 0, respectively, and let R denote a full set of representatives for k in K . Thus R is a subset of the closed unit disc $B_0(1)$ of K which satisfies

- (i) if $r_1, r_2 \in R, r_1 \neq r_2$ then $|r_1 - r_2| = 1$;
- (ii) for every $A \in B_0(1)$ there is an $r \in R$ such that $|A - r| < 1$. In the case $K = \mathbf{Q}_p$ the set $R = \{0, 1, \dots, p-1\}$ satisfies (i) and (ii) above. In general R may be an infinite set.

With the above notation and assuming for simplicity that $0 \in R$, it can be shown (Theorem 12.1, Schikhof [4]) that every $A \in K$ has a unique series representation

$$A = \sum_{j=-n}^{\infty} a_j \pi^j, \quad \text{some } n \in \mathbf{N}, a_j \in R.$$

Consider now the subset $\mathcal{P}_1$ of K consisting of all elements in the 'open' unit ball $B_1(1^-)$, that is those elements of K having series expansion of the form

$$A = 1 + \sum_{j=1}^{\infty} a_j \pi^j, a_j \in R.$$

In an entirely analogous way to the case $K = \mathbb{Q}_p$ we can now prove:

(5.1) THEOREM. *Every $A \in \mathcal{P}_1$, with $A \neq 1$, has a unique convergent product representation (relative to ρ) of the form*

$$A = \prod_{n=1}^{\infty} (1 + b_n \pi^{r_n}),$$

where $0 \neq b_n \in R$, $r_n \in \mathbb{N}$ and $r_{n+1} > r_n$. Further, $\nu(A - P_n) \geq r_{n-1} > n$, where $P_n = (1 + b_1 \pi^{r_1}) \cdots (1 + b_n \pi^{r_n})$.

Other earlier results, such as the Euler-type identity in the proof of Proposition 4.1 for example, also extend to the present setting. One case of special relevance to complex analysis is treated by us more fully elsewhere [2].

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