REES ALGEBRAS AND MIXED MULTIPLICITIES

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To Professor David Rees, in honor of his seventieth birthday

ABSTRACT. Let (R, m) be a local ring of positive dimension d and I and J two m -primary ideals of R . Let T denote the Rees algebra $R[Jt]$ localized at the maximal homogeneous ideal (m, Jt) . It is proved that

$$e((I, Jt)T) = e_0(I|J) + e_1(I|J) + \cdots + e_{d-1}(I|J),$$

where $e_i(I|J)$, $i = 0, 1, \dots, d-1$ are the first d mixed multiplicities of I and J . A formula due to Huneke and Sally concerning the multiplicity of the Rees algebra (of a complete zero-dimensional ideal of a two dimensional regular local ring) at its maximal homogeneous ideal is recovered.

1. Introduction. Bhattacharya [B] and Risler and Teissier [T] obtained a useful and natural generalization of the Hilbert function and multiplicity of an ideal. To elaborate, let I and J be m -primary ideals of a local ring (R, m) of positive dimension d . Let l denote length and $\mathbf{N}$ denote the set of positive integers. Then the function $B(r, s): \mathbf{N}^2 \rightarrow \mathbf{N}$ defined as $B(r, s) = l(R/I^r J^s)$ is a polynomial of total degree d in r and s for large values of r and s . This polynomial is called the Bhattacharya polynomial of I and J . The terms of total degree d in the Bhattacharya polynomial of I and J have the form

$$\frac{1}{d!} \left\{ e_0(I|J)r^d + \cdots + \binom{d}{i} e_i(I|J)r^{d-i}s^i + \cdots + e_d(I|J)s^d \right\}.$$

Here $e_0(I|J), \dots, e_d(I|J)$ are all positive integers, called the mixed multiplicities of I and J . Rees [R1] observed that $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$ where e denotes multiplicity. A fundamental contribution of Risler and Teissier is to identify $e_i(I|J)$ as the multiplicity of an ideal generated by $d-i$ elements from I and i elements from J chosen "sufficiently generally." Rees has reinterpreted the phrase "sufficiently generally" by introducing the concepts of joint reductions [R4] and independent set of general elements of a collection of ideals [R3].

Let $R[Jt]$ denote the Rees algebra of the ideal J , N denote its maximal homogeneous ideal (m, Jt) and set $T = R[Jt]_N$. The purpose of this paper is to prove that

$$e((I, Jt)T) = e_0(I|J) + \cdots + e_{d-1}(I|J).$$

In particular,

$$e(T) = e_0(m|J) + \cdots + e_{d-1}(M|J).$$

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In §2, we recall the necessary definitions and results. §3 is devoted to the proof of the main theorem. In §4 we specialize to two-dimensional regular local rings and recover a formula due to Huneke and Sally [HS] which originally inspired our main theorem. Finally we present an example in §5 which illustrates our main theorem.

2. Preliminaries.

(2.1) By a result due to Valla [V], $\dim R[Jt] = d + 1$ provided $d > 0$.

(2.2) It follows from the Bhattacharya polynomial of I and J that for all positive integers r and s ,

$$e(I^r J^s) = r^d e(I) + \cdots + \binom{d}{i} e_i(I|J) r^{d-i} s^i + \cdots + s^d e(J).$$

In particular, by putting $r = 1 = s$ we get

$$e(IJ) = e(I) + \cdots + \binom{d}{i} e_i(I|J) + \cdots + e(J).$$

If $J = I$ then for all positive integers r and s

$$e(I^{r+s}) = r^d e(I) + \cdots + \binom{d}{i} e_i(I|I) r^{d-i} s^i + \cdots + s^d e(I).$$

Therefore, it follows that $e_i(I|I) = e(I)$ for all i .

(2.3) Recall that an ideal $K \subset I$ is called a reduction of I if there is a positive integer n such that $KI^n = I^{n+1}$ [NR]. It easily follows that $e(K) = e(I)$. Moreover, if L is a reduction of J then $K^r L^s$ is a reduction of $I^r J^s$ for all positive integers r and s . Thus

$$e(K^r L^s) = \sum_{i=0}^d \binom{d}{i} e_i(K|L) r^{d-i} s^i$$

and

$$e(I^r J^s) = \sum_{i=0}^d \binom{d}{i} e_i(I|J) r^{d-i} s^i.$$

Since $e(K^r L^s) = e(I^r J^s)$, we conclude that $e_i(I|J) = e_i(K|L)$.

(2.4) Let m, n be positive integers. Then for all positive integers r and s ,

$$e(I^{rm} J^{sn}) = \sum_{i=0}^d \binom{d}{i} e_i(I|J) (rm)^{d-i} (sn)^i.$$

We also have

$$e(I^{rm} J^{sn}) = \sum_{i=0}^d \binom{d}{i} e_i(I^m|J^n) r^{d-i} s^i.$$

Comparing the above two expressions we get

$$e_i(I^m|J^n) = m^{d-i} n^i e_i(I|J).$$

(2.5) An element $x \in R$ is said to be integral over an ideal I of R if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \quad \text{where } a_i \in I^i \quad \text{for } i = 1, \dots, n.$$

Equivalently, x is integral over I if and only if for some n , $I(I, x)^{n-1} = (I, x)^n$. In other words, I is a reduction of (I, x) . It follows from this fact that an ideal $I \subset J$ is a reduction of an ideal J if and only if every element of J is integral over I . Elements of R which are integral over I form an ideal which we denote by I_a . An ideal I is called a complete ideal if $I = I_a$.

(2.6) LEMMA. Let m and n be positive integers with $m \leq n$. Let δ_{mn} denote the Kronecker δ -function. Then

$$\sum_{k=m}^n \binom{n}{k} \binom{k}{m} (-1)^k = (-1)^m \delta_{mn}.$$

PROOF. If $m = n$ then both sides are clearly equal. Suppose $m < n$. Then

$$\begin{aligned} \sum_{k=m}^n \binom{n}{k} \binom{k}{m} (-1)^k &= \sum_{k=m}^n \frac{n!}{(n-k)!k!} \cdot \frac{k!}{(k-m)!m!} (-1)^k \\ &= \frac{n!}{(n-m)!m!} \sum_{k=m}^n \frac{(n-m)!(-1)^k}{(n-k)!(k-m)!} \\ &= \binom{n}{m} \sum_{j=0}^{n-m} \frac{(n-m)!(-1)^{j+m}}{(n-j-m)!j!} \\ &= (-1)^m \binom{n}{m} \sum_{j=0}^{n-m} \binom{n-m}{j} (-1)^j \\ &= (-1)^m \binom{n}{m} (1-1)^{n-m} = 0. \end{aligned}$$

(2.7) LEMMA. For $j = 0, 1, 2, \dots, d$

$$\sum_{p=0}^{d-j} \binom{d}{j} \binom{d-j}{p} \frac{(-1)^p(d+1)}{j+p+1} = 1.$$

PROOF.

$$\begin{aligned} \sum_{p=0}^{d-j} \binom{d}{j} \binom{d-j}{p} \frac{(-1)^p(d+1)}{j+p+1} &= \sum_{p=j}^d \binom{d}{j} \binom{d-j}{p-j} \frac{(d+1)(-1)^{p-j}}{p+1} \\ &= \sum_{p=j}^d \frac{d!(d-j)!p!(d+1)(-1)^{p-j}}{(d-j)!j!(d-p)!(p-j)!p!(p+1)} \\ &= \sum_{p=j}^d \binom{d+1}{p+1} \binom{p}{j} (-1)^{p-j} \\ &= \sum_{p=0}^d \binom{d+1}{p+1} \binom{p}{j} (-1)^{p-j}. \end{aligned}$$

For $j = 0$ the above sum reduces to

$$\sum_{p=0}^d \binom{d+1}{p+1} (-1)^p = 1 - (1-1)^{d+1} = 1.$$

Suppose that we have proved that the equality is true for $j = 0, 1, \dots, q-1$. Then we show it for $j = q$

$$\begin{aligned} & \sum_{p=0}^d \binom{d+1}{p+1} \binom{p}{q} (-1)^{p-q} \\ &= \sum_{p=0}^d \binom{d+1}{p+1} \binom{p+1}{q} (-1)^{p-q} + \sum_{p=0}^d \binom{d+1}{p+1} \binom{p}{q-1} (-1)^{p-(q-1)}. \end{aligned}$$

The first sum in the above equation is zero by Lemma 2.6 and the second sum is equal to 1 by the induction hypothesis.

(2.8) LEMMA. For positive integers n and r ,

$$1^r + 2^r + \dots + n^r = \frac{n^{r+1}}{r+1} + \frac{1}{2}n^r + \dots.$$

PROOF (R. SIVARAMAKRISHNAN). Write $S_r(n) = 1^r + 2^r + \dots + n^r$. By [Ri, p. 159] we have the identity

$$S_r(n) - \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} S_k(n) = n^r.$$

Change r to $r+1$ to obtain

$$S_{r+1}(n) - \left\{ S_{r+1}(n) - \binom{r+1}{r} S_r(n) + \binom{r+1}{r-1} S_{r-1}(n) + \dots \right\} = n^{r+1}.$$

Hence

$$\binom{r+1}{1} S_r(n) - \binom{r+1}{2} S_{r-1}(n) + \dots = n^{r+1}.$$

The lemma is trivial for $r = 1$. Assume that it is proved for all positive integers less than or equal to $r-1$. By the induction hypothesis we get

$$(r+1)S_r(n) - \binom{r+1}{2} \left[\frac{n^r}{r} + \frac{1}{2}n^{r-1} + \dots \right] + \dots = n^{r+1}.$$

Thus

$$S_r(n) = \frac{n^{r+1}}{r+1} + \frac{1}{2}n^r + \dots.$$

REMARK. For further discussion of $S_r(n)$, see [Ra, Chapter 1] where $S_r(n)$ is expressed in terms of Bernoulli polynomials.

3. The main theorem.

(3.1) THEOREM. Let (R, m) be a local ring of positive dimension d and I, J m -primary ideals of R . Let N denote the unique maximal homogeneous ideal (m, Jt) of the Rees algebra $R[Jt]$ and set $T = R[Jt]_N$. Then

$$e((I, Jt)T) = e_0(I|J) + \dots + e_{d-1}(I|J).$$

PROOF. Let K denote the ideal $(I, Jt)T$. To compute $e(K)$ we compute $l(T/K^n)$ for large n . The direct sum decompositions of $R[Jt]$ and $(I, Jt)^n$ are

$$\begin{aligned} R[Jt] &= R \oplus Jt \oplus J^2t^2 \oplus \dots \oplus J^{n-1}t^{n-1} \oplus J^n t^n \oplus \dots; \\ (I, Jt)^n &= I^n \oplus I^{n-1}Jt \oplus I^{n-2}J^2t^2 \oplus \dots \oplus IJ^{n-1}t^{n-1} \oplus J^n t^n \oplus \dots. \end{aligned}$$

Therefore,

$$\begin{aligned} l(T/K^n) &= \sum_{i=0}^n l(J^i/I^{n-i}J^i) \\ &= \sum_{i=0}^n l(R/I^{n-i}J^i) - \sum_{i=0}^n l(R/J^i). \end{aligned}$$

Now we make use of the Bhattacharya polynomial of I and J to compute $l(R/I^{n-i}J^i)$. We may assume that the Bhattacharya function of I and J coincides with the Bhattacharya polynomial of I and J , since this assumption does not affect the leading coefficient of the Hilbert polynomial of K . Therefore we obtain, by setting $e_k = e_k(I|J)$ for $k = 0, 1, \dots, d$,

$$\begin{aligned} \sum_{i=0}^n l(R/I^{n-i}J^i) &= \sum_{i=0}^n \sum_{k=0}^d \binom{d}{k} (e_k/d!)(n-i)^{d-k} \cdot i^k + \dots \\ &= \sum_{i=0}^n \sum_{k=0}^d \binom{d}{k} ((e_k) \cdot (i^k)/d!) \sum_{r=0}^{d-k} \binom{d-k}{r} n^{d-k-r} i^r (-1)^r + \dots \\ &= \sum_{k=0}^d \sum_{r=0}^{d-k} \binom{d}{k} \binom{d-k}{r} (e_k/d!) n^{d-k-r} (-1)^r \sum_{i=0}^n i^{k+r} + \dots \\ &= \sum_{k=0}^d \sum_{r=0}^{d-k} \binom{d}{k} \binom{d-k}{r} (e_k/d!) (-1)^r \frac{n^{d+1}}{k+r+1} + \dots \\ &= \frac{n^{d+1}}{(d+1)!} \sum_{k=0}^d \left\{ \sum_{r=0}^{d-k} \binom{d}{k} \binom{d-k}{r} \frac{(-1)^r (d+1)}{k+r+1} \right\} e_k + \dots \\ &= \frac{n^{d+1}}{(d+1)!} (e_0 + e_1 + \dots + e_d) + \dots \end{aligned}$$

Therefore for large n

$$\begin{aligned} l(T/K^n) &= \frac{n^{d+1}}{(d+1)!} (e_0 + e_1 + \dots + e_d) - \sum_{i=0}^n \frac{i^d}{d!} e(J) + \dots \\ &= \frac{n^{d+1}}{(d+1)!} (e_0 + e_1 + \dots + e_d) - \left(\frac{n^{d+1}}{(d+1)!} e(J) \right) + \dots \\ &= \frac{n^{d+1}}{(d+1)!} (e_0 + e_1 + \dots + e_{d-1}) + \dots \end{aligned}$$

Hence $e(K) = e_0 + e_1 + \dots + e_{d-1}$ and the proof is complete.

(3.2) COROLLARY. Let (R, m) be a local ring of positive dimension d . Let r be a positive integer. Then by setting $N = (m, m^r t)$ we have

$$e(R[m^r t]_N) = e(R)(1 + r + \dots + r^{d-1}).$$

PROOF. By the theorem

$$\begin{aligned} e(R[m^r t]_N) &= e_0(m|m^r) + e_1(m|m^r) + \dots + e_{d-1}(m|m^r) \\ &= e(R) + r e_1(m|m) + \dots + r^{d-1} e_{d-1}(m|m) \\ &= e(R)(1 + r + \dots + r^{d-1}). \end{aligned}$$

4. The case of two-dimensional regular local rings. Throughout this section (R, m) will denote a two-dimensional regular local ring. Let o denote the valuation of the quotient field of R such that for $0 \neq x \in R$

$$o(x) = \max\{n \mid x \in m^n\}.$$

For an ideal I of R , put

$$o(I) = \min\{o(x) \mid 0 \neq x \in I\}.$$

Let N denote the unique maximal homogeneous ideal of the Rees algebra $R[It]$. Huneke and Sally [HS] proved that if I is an m -primary complete ideal, then

$$e(R[It]_N) = 1 + o(I).$$

This formula originally inspired our main theorem. We recover it as a consequence. First we need to recall some results concerning complete and contracted ideals in a two-dimensional regular local ring. An ideal I of R is called a contracted ideal [ZS, Appendix 5] if there is an $x \in m \setminus m^2$ such that $IR[m/x] \cap R = I$. Zariski [ZS, Appendix 5] proved that a complete m -primary ideal is contracted, a product of contracted ideals is contracted, and a product of complete ideals is complete. The theory of complete ideals in two-dimensional regular local rings was founded by Zariski in [Z].

Lipman [L] and Rees [R2] proved that if I is an m -primary contracted ideal of R then $\mu(I)$ = minimum number of generators for $I = 1 + o(I)$. Huneke [H] proved that the converse is true provided R/m is infinite. Now we are ready to prove

(4.1) **THEOREM (HUNEKE-SALLY).** *Let (R, m) be a two-dimensional regular local ring. For an m -primary ideal I of R , let N denote the unique maximal homogeneous ideal (m, It) of the Rees algebra $R[It]$. Then $e(R[It]_N) = \mu(I_a) = 1 + o(I)$.*

PROOF. By the main theorem

$$e(R[It]_N) = e_0(m|I) + e_1(m|I) = 1 + e_1(m|I).$$

Since m^n is complete for all n , $o(I) = o(I_a)$. Also, $e_1(m|I) = e_1(m|I_a)$. Therefore we may assume that I is a complete ideal. Hence I is contracted and consequently $I^r m^s$ is a contracted ideal for all positive integers r and s . Therefore $\mu(I^r m^s) = s + ro(I) + 1$.

Now we calculate the Bhattacharya polynomial of I and m . For all r and s ,

$$\begin{aligned} l(R/I^r m^s) &= l(R/I^r) + \sum_{i=0}^{s-1} l(I^r m^i / I^r m^{i+1}) = l(R/I^r) + \sum_{i=0}^{s-1} \mu(I^r m^i) \\ &= l(R/I^r) + \sum_{i=0}^{s-1} (i + ro(I) + 1) = l(R/I^r) + rso(I) + s(s+1)/2 \\ &= l(R/I^r) + rso(I) + l(R/m^s) = \frac{1}{2} \{e(I)r^2 + 2rso(I) + s^2 e(m)\} + \cdots \end{aligned}$$

(by [H, Theorem 7.2]). Therefore $e_1(m|I) = o(I)$ which finishes the proof.

5. An example. Let (R, m) be a three-dimensional regular local ring with (x, y, z) a regular system of parameters, i.e., $m = (x, y, z)$. Let J and I denote

the ideals (x, y^2, yz, z^2) and mJ respectively. We show that $e(R[Jt]_N) = 4$ where $N = (m, Jt)$. Set $e_i(m|J) = e_i$ for $i = 0, 1, 2, 3$. By the main theorem

$$e(R[Jt]_N) = e_0 + e_1 + e_2 = 1 + e_1 + e_2.$$

On the other hand,

$$\begin{aligned} e(mJ) &= e_0(m|J) + \binom{3}{1} e_1(m|J) + \binom{3}{2} e_2(m|J) + e_3(m|J) \\ &= 1 + 3(e_1 + e_2) + e(J). \end{aligned}$$

Since yz is integral over J , $e(J) = e((x, y^2, z^2))$. Since R is Cohen-Macaulay

$$e((x, y^2, z^2)) = l(R/(x, y^2, z^2)) = 4.$$

Therefore, $e(mJ) = 5 + 3(e_1 + e_2)$. We now calculate the multiplicity of $I = mJ$ by calculating its Hilbert polynomial. For all n ,

$$\begin{aligned} I^n &= (x^{2n}, \dots, x^{2n-i}(y, z)^i, \dots, x^n(y, z)^n, \\ &\quad x^{n-1}(y, z)^{n+2}, \dots, x^{n-i}(y, z)^{n+2i}, \dots, (y, z)^{3n}). \end{aligned}$$

Therefore, $l(R/I^n)$ = the number of monomials of the form $x^i(y^j z^k)$ for $i = 0, 1, \dots, 2n - 1$ not belonging to I^n . An easy counting argument shows that

$$l(R/I^n) = \sum_{i=1}^n \binom{i+1}{2} + \sum_{i=1}^n \binom{n+2i+1}{2}.$$

From this we obtain

$$\begin{aligned} l(R/I) &= \binom{2}{2} + \binom{4}{2} = 7, \quad l(R/I^2) = \binom{2}{2} + \binom{3}{2} + \binom{5}{2} + \binom{7}{2} = 35, \\ l(R/I^3) &= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{6}{2} + \binom{8}{2} + \binom{10}{2} = 98, \\ l(R/I^4) &= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{7}{2} + \binom{9}{2} + \binom{11}{2} + \binom{13}{2} = 210. \end{aligned}$$

The above expression for $l(R/I^n)$ shows that the Hilbert polynomial of I coincides with the Hilbert function of I . Write the Hilbert polynomial in the standard form

$$l(R/I^n) = f_0 \binom{n+2}{3} - f_1 \binom{n+1}{2} + f_2 \binom{n}{1} - f_3.$$

Substitute $n = 1, 2, 3, 4$ to obtain the following system of linear equations:

$$\begin{aligned} f_0 - f_1 + f_2 - f_3 &= 7, \\ 4f_0 - 3f_1 + 2f_2 - f_3 &= 35, \\ 10f_0 - 6f_1 + 3f_2 - f_3 &= 98, \\ 20f_0 - 10f_1 + 4f_2 - f_3 &= 210. \end{aligned}$$

It turns out that $f_0 = 14$, $f_1 = 7$, and $f_2 = f_3 = 0$. Thus the Hilbert polynomial of I is given by

$$l(R/I^{n+1}) = 14 \binom{n+3}{3} - 7 \binom{n+2}{2} \quad \text{for all } n = 0, 1, 2, \dots$$

Therefore $e(mJ) = 14 = 1 + 3(e_1 + e_2) + 4$. This gives that $e_1 + e_2 = 3$ and consequently $e(R[Jt]_N) = e_0 + e_1 + e_2 = 4$.

Alternatively, we can calculate e_1 and e_2 by using joint reductions. Rees [R4] defines these as follows: Let (R, m) be a d -dimensional local ring and $I_1, I_2, \dots, I_d$ be ideals of R . Let $R = (r_1, \dots, r_d)$ be a set of nonnegative integers. Denote by I^R the ideal $I_1^{r_1} \cdots I_d^{r_d}$. Let $R(i)$ denote the set of integers $(r_1, \dots, r_i - 1, \dots, r_d)$. Then a set of elements $x_1, \dots, x_d$ such that $x_i \in I_i$ for $i = 1, 2, \dots, d$ is called a joint reduction of $I = (I_1, \dots, I_d)$ if for some R

$$I^R = x_1 I^{R(1)} + x_2 I^{R(2)} + \cdots + x_d I^{R(d)}.$$

THEOREM (REES [R4]). *Let I and J be m -primary ideals of a d -dimensional local ring. Then $e_i(I|J)$, $i = 0, 1, \dots, d$ is the multiplicity of the ideal generated by any joint reduction of the set of ideals which consists of $d - i$ copies of I and i copies of J .*

Since $K = (x, y^2, z^2)$ is a reduction of $J = (x, y^2, yz, z^2)$, $e_i(m|J) = e_i(m|K)$ for all i . The equation

$$m^2 K^2 = ymK^2 + zmK^2 + xm^2 K$$

shows that (y, z, x) is a joint reduction of the set (m, m, K) . Therefore $e_1(m|J) = e(y, z, x) = 1$. The equation

$$m^2 K^2 = ymK^2 + xm^2 K + z^2 m^2 K$$

shows that (y, x, z^2) is a joint reduction of the set (m, K, K) . Therefore $e_2(m|J) = e(y, x, z^2) = 2$. Consequently, $e(R[Jt]_N) = e_0 + e_1 + e_2 = 1 + 1 + 2 = 4$.

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REFERENCES

- [B] P. B. Bhattacharya, *The Hilbert function of two ideals*, Proc. Cambridge Philos. Soc. **53** (1957), 568–575.
- [H] C. Huneke, *Complete ideals in two dimensional regular local rings (after Zariski and Lipman)*, Proc. "Microprogram in Commutative Algebra" (MSRI, Berkeley, Calif., 1987) (to appear).
- [HS] C. Huneke and J. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra **115** (1988), 481–500.
- [L] J. Lipman, *On complete ideals in regular local rings*, preprint, 1987.
- [NR] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.
- [R1] D. Rees, *$\mathcal{A}$ -transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17.
- [R2] ———, *Hilbert functions and pseudo-rational local rings of dimension two*, J. London Math. Soc. **24** (1981), 467–479.

- [R3] ———, *The general extension of a local ring and mixed multiplicities*, Algebra, Algebraic Topology and Their Interactions (Proc., Stockholm, 1983), Lecture Notes in Math., vol. 1183, Springer-Verlag, Berlin, 1986, pp. 339–360.
- [R4] ———, *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc. **29** (1984), 397–414.
- [Ra] H. Rademacher, *Topics in analytic number theory*, Springer-Verlag, New York, 1973.
- [Ri] J. Riordan, *Combinatorial identities*, Wiley, New York, 1968.
- [T] B. Teissier, *Cycles évanescents, sections planes, et conditions de Whitney*, Singularités à Car-gèse, 1972, Astérisque 7-8 (1973), 285–362.
- [V] G. Valla, *Certain graded algebras are always Cohen-Macaulay*, J. Algebra **42** (1976), 537–548.
- [Z] O. Zariski, *Polynomial ideals defined by infinitely near base points*, Amer. J. Math. **60** (1938), 151–204.
- [ZS] O. Zariski and P. Samuel, *Commutative algebra*, Vol. II, Van Nostrand, Princeton, N. J., 1960.

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