SEMICONFLUENT MAPS AND CONTINUA CONTAINING NO n -ODS

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ABSTRACT. J. F. Davis proved that the semiconfluent image of an atriodic continuum is atriodic and asked if this result could be generalized to n -ods. In this paper the question is answered in the affirmative. It is proved that the semiconfluent image of a Hausdorff continuum containing no n -ods must contain no n -ods

In [2, Theorem 1, p. 152] E. E. Grace and the author proved that the semiconfluent image of a chainable continuum is atriodic. Chainable continua are atriodic, and recently J. F. Davis [1, Theorem 1, p. 579] proved the more general result that the semiconfluent image of an atriodic continuum is atriodic. In [1], Davis raised the general question as to whether the semiconfluent image of a continuum that contains no n -ods must contain no n -ods. It is the purpose of this paper to give an affirmative answer to this question for Hausdorff continua. This result cannot be extended to weakly confluent maps in any way.

Definitions and preliminary results. A *continuum* is a compact connected Hausdorff space and a *map* is a continuous function. A map f of a continuum X onto a continuum Y is *confluent* (*weakly confluent*) if, for each subcontinuum K of Y , each (some) component of $f^{-1}(K)$ is mapped by f onto K . A map is *semiconfluent* if, for each subcontinuum K of Y and each two components C_1 and C_2 of $f^{-1}(K)$, either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. By definition, a confluent map is semiconfluent, and T. Maćkowiak has proved [3, Corollary 3.2, p. 254] that a semiconfluent map is weakly confluent. A continuum is an n -od, $n \geq 3$, if it contains a subcontinuum whose complement has at least n components. The main result of this paper is the following theorem.

THEOREM. *If f is a semiconfluent map of a continuum X that contains no n -ods onto a continuum Y , then Y contains no n -ods.*

In order to prove this theorem two lemmas are needed.

LEMMA 1. *Suppose $A_1, \dots, A_n$, $n \geq 2$, are mutually disjoint closed subsets of a continuum Y , f is a semiconfluent map of a continuum X onto Y and K is a subcontinuum of Y intersecting each A_j , $1 \leq j \leq n$. Then there exists i , $1 \leq i \leq n$, such that if C is a component of $f^{-1}(K)$ where $C \cap f^{-1}(A_j) \neq \emptyset$ for some j , $1 \leq j \leq n$, then $C \cap f^{-1}(A_i) \neq \emptyset$.*

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PROOF. For $1 \leq j \leq n$, let $V_j = f^{-1}(A_j)$. Every component of $f^{-1}(K)$ that intersects V_1 also intersects V_2 or every component of $f^{-1}(K)$ that intersects V_2 also intersects V_1 . For if C_1 and C_2 are components of $f^{-1}(K)$ such that $C_1 \cap V_1 \neq \emptyset \neq C_2 \cap V_2$ and $C_1 \cap V_2 = \emptyset = C_2 \cap V_1$, then $f(C_1) \not\subset f(C_2)$ and $f(C_2) \not\subset f(C_1)$, contradicting the semiconfluence of f . Without loss of generality assume that every component of $f^{-1}(K)$ that intersects V_2 also intersects V_1 . Suppose that for $2 \leq j \leq j_0 < n$ every component of $f^{-1}(K)$ that intersects V_j also intersects V_1 . By the above argument, either every component of $f^{-1}(K)$ that intersects V_{j_0+1} also intersects V_1 or conversely. In the first case every component of $f^{-1}(K)$ that intersects any one of $V_2, \dots, V_{j_0+1}$ also intersects V_1 . In the converse case, any component of $f^{-1}(K)$ that intersects any one of $V_2, \dots, V_{j_0}$ also intersects V_1 and, hence, must intersect V_{j_0+1} . By induction, the lemma is established.

The next lemma is an easy generalization of [4, Theorem 48, p. 17] and is stated without proof.

LEMMA 2. *Let $C_1, \dots, C_k$, $k \geq 2$, be mutually disjoint subcontinua of a continuum M and M_0 be a subcontinuum of M irreducible about $\bigcup_{i=1}^k C_i$. Then $M_0 \setminus \bigcup_{i=1}^k C_i$ has at most $k - 1$ components.*

Proof of the main result. Using the lemmas the theorem stated earlier is proved.

PROOF. Suppose Y contains an n -od, i.e., there exist $H_1, \dots, H_n, Q$, subcontinua of Y , such that $H_i \setminus Q \neq \emptyset$, $1 \leq i \leq n$ and $H_i \cap H_j = Q$, $1 \leq i < j \leq n$. Let $U_1, \dots, U_n$ be open sets with mutually disjoint closures such that $U_i \cap H_i \neq \emptyset$ but $\text{cl}(U_i)$ intersects no other H_j , $1 \leq j \leq n$. For $1 \leq i \leq n$, let K_i be the component of $H_i \setminus U_i$ that contains Q . Observe that $K_i \cap \text{cl}(U_i) \neq \emptyset$, $1 \leq i \leq n$ [4, Theorem 50, p. 18]. Let $H = \bigcup_{i=1}^n H_i$ and $K = \bigcup_{i=1}^n K_i$ and note that K is the component of $H \setminus \bigcup_{i=1}^n U_i$ that contains Q . Also, for $1 \leq i \leq n$, let $V_i = f^{-1}(\text{cl}(U_i))$. By Lemma 1, there exists an integer i such that every component of $f^{-1}(K)$ that intersects some V_j , $1 \leq j \leq n$, also intersects V_i . Assume without loss of generality that $V_i = V_n$. Let $H' = \bigcup_{i=1}^{n-1} H_i$ and $K' = \bigcup_{i=1}^{n-1} K_i$. Analogous to H and K , note that K' is the component of $H' \setminus \bigcup_{i=1}^{n-1} U_i$ that contains Q .

Semiconfluent maps are weakly confluent. Hence there exists a component M of $f^{-1}(H')$ such that $f(M) = H'$.

Each component of $f^{-1}(K')$ that lies in M intersects V_i for some i , $1 \leq i \leq n-1$. To see this assume C is a component of $f^{-1}(K')$ in M such that $C \cap V_i = \emptyset$, $1 \leq i \leq n-1$, and let R be an open set in M such that $C \subset R \subset \text{cl}(R) \subset M \setminus \bigcup_{i=1}^{n-1} V_i$. Let S be a subcontinuum of $\text{cl}(R)$ such that $C \subset S$ and $S \cap \text{Bd}(R) \neq \emptyset$ [4, Theorem 50, p. 18]. Then $f(S) \subset K'$ since $f(S) \subset H'$ and $f(S) \cap K' \neq \emptyset$. This is impossible since S properly contains C and C is a component of $f^{-1}(K')$. Next we show there are at most $n-1$ components of $f^{-1}(K')$ that lie in M . Suppose there are at least n such components and let $C_1, \dots, C_n$ be n of them. There exist open sets $R_1, \dots, R_n$ in X with mutually disjoint closures such that $C_i \subset R_i$ and $V_n \cap \text{cl}(R_i) = \emptyset$, $1 \leq i \leq n$. Each C_i , $1 \leq i \leq n$, is contained in a component of $f^{-1}(K)$ that intersects V_j for some j , $1 \leq j \leq n-1$. By our previous assumption this component must intersect V_n and, hence, must also intersect $\text{Bd}(R_i)$. So for each i , $1 \leq i \leq n$, there exists a continuum S_i in $f^{-1}(K)$ such that $C_i \subset S_i \subset \text{cl}(R_i)$ and $S_i \cap \text{Bd}(R_i) \neq \emptyset$. Clearly,

$S_i \not\subset M$, $1 \leq i \leq n$, for otherwise, as in the argument above, C_i could not be a component of $f^{-1}(K')$. But $(S_i \setminus M) \cap (S_j \setminus M) = \emptyset$, $1 \leq i < j \leq n$. So $M \cup \bigcup_{i=1}^n S_i$ is an n -od in X , a contradiction.

Let $C_1, \dots, C_k$, $1 \leq k \leq n-1$, be the components of $f^{-1}(K')$ in M and let M_0 be a subcontinuum of M irreducible about $\bigcup_{i=1}^k C_i$ (if $k=1$, $M_0 = C_1$). By Lemma 2, if $k > 1$, $L = M_0 \setminus \bigcup_{i=1}^k C_i$ has at most $k-1$ components.

If $k=1$, define $p=0$. If $1 < k \leq n-1$, let $L_1, \dots, L_p$ be the components of L where $1 \leq p \leq k-1$. Then L_j , $1 \leq j \leq p$, intersects at most one V_i , $1 \leq i \leq n-1$, since $f(L_j)$ is a connected subset of $H' \setminus K'$ and therefore can intersect at most one of the sets $\text{cl}(U_1), \dots, \text{cl}(U_{n-1})$. Then, since $C_1, \dots, C_k$ are components of $f^{-1}(K')$ and subsets of M , for each j , $1 \leq j \leq p$, L_j intersects exactly one V_i , $1 \leq i \leq n-1$. Renumbering, if necessary, let $V_{n-1}, \dots, V_{n-p}$, each have nonvoid intersection with $\bigcup_{j=1}^p L_j$.

For each i , $1 \leq i \leq n-p-1$, let M_i be a subcontinuum of M irreducible from a point of $\text{Int}(V_i)$ to M_0 . For $1 \leq i < j \leq n-p-1$, $(M_i \setminus M_0) \cap (M_j \setminus M_0) = \emptyset$ since $M_i \setminus M_0$ and $M_j \setminus M_0$ are connected [4, Theorem 48, p. 17] and $f(M_i \setminus M_0)$ and $f(M_j \setminus M_0)$ are in different separated subsets of $H' \setminus K'$. As in an earlier part of the proof, let $R_1, \dots, R_k$ be open sets in X with mutually disjoint closures such that $C_i \subset R_i$ and $V_n \cap \text{cl}(R_i) = \emptyset$, $1 \leq i \leq k$. Each C_i , $1 \leq i \leq k$, is contained in a component of $f^{-1}(K)$ that intersects V_j for some j , $1 \leq j \leq n-1$. This component must intersect V_n by our previous assumption, and so must intersect $\text{Bd}(R_i)$. Then for each i , $1 \leq i \leq k$, there exists a continuum S_i in $f^{-1}(K)$ such that $C_i \subset S_i \subset \text{cl}(R_i)$ and $S_i \cap \text{Bd}(R_i) \neq \emptyset$. Clearly $(S_i \setminus M_0) \cap (S_j \setminus M_0) = \emptyset$, $1 \leq i < j \leq k$, since $(S_i \setminus \bigcup_{l=1}^k C_l) \cap (S_j \setminus \bigcup_{l=1}^k C_l) = \emptyset$ and $\bigcup_{l=1}^k C_l \subset M_0$. Also $(S_i \setminus M_0) \cap (M_j \setminus M_0) = \emptyset$, $1 \leq i \leq k$ and $1 \leq j \leq n-p-1$ since $f(S_i \setminus M_0) \subset K$ and $f(M_j \setminus M_0) \subset H' \setminus K' = H' \setminus K$. So $M_0 \cup (\bigcup_{i=1}^k S_i) \cup (\bigcup_{i=1}^{n-p-1} M_i)$ is a $(k+n-p-1)$ -od in X . Since $k+n-p-1 \geq k+n-(k-1)-1 = n$, X contains an n -od. But this contradicts the hypothesis that X contains no n -ods. Hence the theorem is true.

REMARKS. E. E. Grace and the author [2, Example 1, p. 157] have an example of a chainable plane continuum and a weakly confluent map of it onto a simple triod. It is fairly straightforward to alter this example in R^n for any n to obtain a chainable continuum and a weakly confluent map of it onto an $(n+1)$ -od. Since a chainable continuum contains no n -ods, $n \geq 3$, this shows that there is no way to generalize the theorem from semiconfluent maps to weakly confluent maps for any combination in which the domain contains no n -ods and the range contains no m -ods, $m \geq n$.

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Added in proof. W. Bula in a private conversation with the author mentioned that if $n=2$ is allowed in the main theorem, then the following result, due to J. F. Davis, is an immediate corollary: If f is a semiconfluent map of a continuum X that is hereditarily indecomposable onto a continuum Y , then Y is hereditarily indecomposable.

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