FINDING OCTONION ALGEBRAS IN ASSOCIATIVE ALGEBRAS

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ABSTRACT. An embedding of octonion algebras in central simple degree three associative algebras is given.

Using standard operations on trace 0 elements of a simple 9-dimensional associative algebra (such as 3×3 matrices) over a field of characteristic not 3 containing a primitive cube root of 1, we obtain in the Theorem the somewhat surprising embedding of an octonion algebra in an associative algebra. One clue to this embedding is that the Tits construction in [4] applied to $F \oplus \mathcal{O}$ ($\mathcal{O}$ split octonion) or to F_3 (3×3 matrices) gives the same 27-dimensional exceptional Jordan algebra, suggesting that $F \oplus \mathcal{O}$ is related to F_3 . A similar embedding of the 27-dimensional exceptional Jordan algebra $\mathcal{J}$ as trace 0 elements in the degree 4 special Jordan algebra $\mathcal{H}_4$ of 4×4 hermitian quaternionic matrices was previously known (see [1 and 2]). In this case, a relation between $F \oplus \mathcal{J}$ and $\mathcal{H}_4$ was suggested by the Cayley-Dickson process in [1] which applied to $F \oplus \mathcal{J}$ or to $\mathcal{H}_4$ gives the same 56-dimensional structurable algebra.

Our approach is to show that the set A_0 of trace 0 elements of a separable degree 3 alternative algebra A carries a nondegenerate quadratic form S permitting composition $S(x * y) = S(x)S(y)$, from which it follows that an isotope of $(A_0, *)$ is a unital composition algebra. If A has dimension 9, then A_0 has dimension 8, and the composition algebra must be octonion.

We begin by establishing some basic facts about degree 3 alternative algebras.

LEMMA. *Let A be a finite-dimensional alternative algebra of degree 3, with generic minimum polynomial $p_x(\lambda) = \lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x)1$. Setting $x^\# = x^2 - T(x)x + S(x)1$, $S(x, y) = S(x+y) - S(x) - S(y)$, and $x \times y = (x+y)^\# - x^\# - y^\#$, we have*

- (a) $N(xy) = N(x)N(y)$,
- (b) $S(1) = T(1) = 3$, $N(1) = 1$,
- (c) $S(x) = T(x^\#)$,
- (d) $S(x, y) = T(x \times y)$,
- (e) $S(x, 1) = 2T(x)$,
- (f) $T(x)T(y) = T(xy) + T(x \times y)$,
- (g) $2S(x) = T(x)^2 - T(x^2)$,
- (h) $T(xy) = T(yx)$,
- (i) $x \times y = xy + yx - T(x)y - T(y)x + (T(x)T(y) - T(xy))1$,
- (j) $x^{\#\#} = N(x)x$,

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- (k) $(x \times y)^\# + x^\# \times y^\# = T(x^\# y) y + T(x y^\#) x$,
- (l) $(xy)^\# = y^\# x^\#$,
- (m) $S(xy) = S(yx)$.

PROOF. We may assume the field is infinite and use differential calculus as in [5] where (a), (b) and $p_x(\lambda) = N(\lambda 1 - x)$ are proved. We have $x^{-1} = N(x)^{-1} x^\#$ if $N(x) \neq 0$, since $xx^\# - N(x)1 = x^\# x - N(x)1 = p_x(x) = 0$. Since $N(x + \lambda y) = \lambda^3 N(y) + \lambda^2 \partial_x N|_y + \lambda \partial_y N|_x + N(x)$, we see $T(x) = \partial_x N|_1$ and $S(x) = \partial_1 N|_x$. Since $N(x + \lambda y) = N(xy^{-1} + \lambda 1)N(y)$, we also see $\partial_x N|_y = T(xy^{-1})N(y) = T(xy^\#)$, if $N(x) \neq 0$. By density, $\partial_x N|_y = T(xy^\#)$. Now $x = 1$ gives (c), and (d) follows by linearization. We see $x \times y = \partial_x \#|_y$ so $T(x(y \times z)) = \partial_y \partial_x N|_z$ is symmetric in x, y , and z . In particular, $S(x, 1) = \partial_x S|_1 = \partial_x \partial_1 N|_1 = T(x(1 \times 1)) = 2T(x)$ showing (e), since $1 \times 1 = 2(1^\#) = 2$ by the definition of $X, \#$, and (b). We also compute $x \times 1 = \partial_x \#|_1 = x1 + 1x - T(x)1 - T(1)x + S(x, 1)1 = T(x)1 - x$. Hence, $T(x \times y) = T((x \times 1)y) = T(T(x)y - xy) = T(x)T(y) - T(xy)$ showing (f). Clearly, (g) and (h) follow from (f). Moreover, (i) follows from (d) and (f). Since $x^{\#\#} = (N(x)x^{-1})^\# = N(x)^2 N(x^{-1})(x^{-1})^{-1}$ and $N(x)N(x^{-1}) = N(1) = 1$, we see (j) holds by density. Moreover, (k) is the coefficient of λ^2 in (j) with x replaced by $x + \lambda y$, using $\partial_x N|_y = T(xy^\#)$. To get (l), note $(xy)^\# = N(xy)(xy)^{-1} = N(x)N(y)y^{-1}x^{-1} = y^\# x^\#$ and use density. Finally, (m) follows from (c), (l), and (h).

Before looking at the general situation, we motivate the definition of $x * y$ in the following Theorem by investigating the relationship between A_0 and $\mathcal{O}$ in the particular degree 3 alternative algebra $A = F \oplus \mathcal{O}$. If $x = (\alpha, a)$, then $T(x) = \alpha + t(a)$, $S(x) = \alpha t(a) + n(a)$, and $N(x) = \alpha n(a)$, in terms of the trace t and the norm n on $\mathcal{O}$. The projection $x \rightarrow a$ restricted to A_0 is an isometry of (A_0, S) with $(\mathcal{O}, n - t^2)$ since $\alpha = -t(a)$ on A_0 . Also, if $\omega + \omega^2 + 1 = 0$ in F , then $a \rightarrow a + \omega t(a)$ is an isometry of $(\mathcal{O}, n - t^2)$ with $(\mathcal{O}, n)$. Combining, we get an isometry $\theta: (-t(a), a) \rightarrow a + \omega t(a)$ of (A_0, S) with $(\mathcal{O}, n)$. Since n admits composition relative to the product in $\mathcal{O}$, S automatically admits composition $S(x * y) = S(x)S(y)$ relative to the product

$$(1) \quad x * y = \theta_1^{-1}(\theta_2(x)\theta_3(y))$$

which has been pulled back from $\mathcal{O}$ by any isometries θ_i of (A_0, S) with $(\mathcal{O}, n)$. In particular, taking all $\theta_i = \theta$ above gives a product $x \square y$ on A_0 isomorphic to that on $\mathcal{O}$. We would like to express $x \square y$ in terms of the original product on A . For $y = (\beta, b) \in A_0$, a direct computation (using $a + \bar{a} = t(a)$) shows

$$(2) \quad \theta(x)\theta(y) = -\omega^2 ab + \omega ba - \gamma,$$

where $\gamma = \omega t(ab) + t(a)t(b)$. On the other hand, if $x \rightarrow x_0$ is the projection onto A_0 , i.e. $x_0 = x - \frac{1}{3}T(x)1$, then we also have

$$(3) \quad \theta((xy)_0) = -\bar{b}\bar{a} + \rho\gamma,$$

where $\rho = (\omega - \omega^2)^{-1}$. Consequently, $\theta((\omega xy - \omega^2 yx)_0) = -\omega \bar{b}\bar{a} + \omega^2 \bar{a}\bar{b} + \gamma = (j\theta(x))(-j\theta(y))$ where $j(a) = \bar{a}$. Taking $\theta_1 = \theta, \theta_2 = j\theta$, and $\theta_3 = -j\theta$ in (1), shows S admits composition relative to $x * y = (\omega xy - \omega^2 yx)_0$. Moreover, if $u \in A_0$ with $\theta(u) = 1$, then $\theta(x * u) = -j\theta(x) = \theta(u * x)$ so $\theta((x * u) * ((-u) * y)) = \theta(x)\theta(y)$ and $x \square y$ is the product in the $(u, -u)$ -isotope of $(A_0, *)$. This suggests the product $*$ used in the Theorem.

THEOREM. *If A is a finite-dimensional alternative algebra of degree 3 over a field F of characteristic not 3 containing an element ω with $\omega^2 + \omega + 1 = 0$, then the coefficient $S(x)$ of λ in the generic minimum polynomial permits composition $S(a * b) = S(a)S(b)$ on $A_0 = \{x: T(x) = 0\}$ relative to the product $a * b = (\omega ab - \omega^2 ba)_0$ where $x_0 = x - \frac{1}{3}T(x)1$. If A is separable over F , then the quadratic form S is nondegenerate, and $(A_0, *)$ is a quasi-composition algebra with a unital isotope which is a composition algebra. If A also has dimension 9, then the isotope is an octonion algebra.*

PROOF. We have $S(a)S(b) = T(a^\#)T(b^\#)$ (by (c))

$$= T(a^\#b^\#) + T(a^\# \times b^\#) \quad (\text{by (f)})$$

$$= T(a^\#b^\#) - T((a \times b)^\#)$$

(by T applied to (k) using $a, b \in A_0$)

$$= S(ab) - S(a \times b) \quad (\text{by (c), (l), and (m)})$$

$$= S(ab) - S(ab + ba - T(ab)1) \quad (\text{by (i)})$$

$$= S(ab) - S(ab) - S(ba) - S(ab, ba) + T(ab)S(ab + ba, 1) \\ - T(ab)^2 S(1)$$

$$= -S(ab) - S(ab, ba) + T(ab)^2$$

(by (b), (m), (e), and (h)).

On the other hand, $S(x_0) = S(x - \frac{1}{3}T(x)1) = S(x) - \frac{1}{3}S(x, 1)T(x) + \frac{1}{9}T(x)^2 S(1) = S(x) - \frac{1}{3}T(x)^2$ by (e) and (b). Thus, $S(a * b) = S(\omega ab - \omega^2 ba) - \frac{1}{3}T(\omega ab - \omega^2 ba)^2 = \omega^2 S(ab) + \omega^4 S(ba) - \omega \omega^2 S(ab, ba) - \frac{1}{3}(\omega - \omega^2)^2 T(ab)^2$ (by (h)) $= -S(ab) - S(ab, ba) + T(ab)^2$, since $\omega^3 = 1, \omega + \omega^2 = -1$, and $(\omega - \omega^2)^2 = -3$. If A is separable over F , then $T(xy)$ is nondegenerate on A and on A_0 . Since $S(x, y) = -T(xy)$ on A_0 by (d) and (f), S is also nondegenerate on A_0 . The remainder now follows from Kaplansky [3]. Indeed, whenever $S(u) \neq 0$, we get a unital composition algebra from the isotope with product $a \square b = (R_u^{*-1}a) * (L_u^{*-1}b)$ with unit $u * u$. Here R_u^* and L_u^* denote the right and left multiplication in $(A_0, *)$.

EXAMPLE. If $A = F_3$ and $u = \text{diag}\{2\rho, -\rho, -\rho\}$ for $\rho = (\omega - \omega^2)^{-1}$, then $(F_3)_0$ is a split octonion algebra with identity element u and norm form $\text{tr}(\text{adj } a)$ relative to the product $a \square b = (u * a) * (b * u)$ where $x * y = \omega xy - \omega^2 yx + \rho \text{tr}(xy) \text{Id}$.

PROOF. We use $A = F_3$ in the Theorem, so $T(x) = \text{tr}(x), N(x) = \det(x), S(x) = \text{tr}(\text{adj } x)$, and $x^\# = \text{adj } x$, the matrix adjoint. Since $\rho^{-2} = -3$, we see $-\frac{1}{3}(\omega - \omega^2) = \rho^2 \rho^{-1} = \rho$ and $x * y = (\omega xy - \omega^2 yx)_0 = \omega xy - \omega^2 yx + \rho \text{tr}(xy) \text{Id}$. Let $e_1 = \text{diag}\{1, \omega, \omega^2\}, e_2 = \text{diag}\{1, \omega^2, \omega\}$, so $e_i \in A_0$ have $e_i e_j = \text{Id}, N(e_i) = 1, e_i^\# = e_i^2 = e_j$ for $i \neq j$. We see $e_i * e_i = (\omega - \omega^2)(e_i^2)_0 = \rho^{-1} e_j, i \neq j$, and $e_i * e_j = (\omega - \omega^2)(e_i e_j)_0 = 0$. If $u = \rho e_1 + \rho e_2 = \text{diag}\{2\rho, -\rho, -\rho\}$, we see $u * e_i = e_i * u = e_j$ and $u * u = u$. If $L_x y = R_y x = x * y$ on A_0 , we note that L_u^* is the inverse of R_u^* on $Fe_1 + Fe_2$. In general, if $L_x y = R_y x = xy$, and if $d = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, then L_d and R_d act on matrices $FE_{ij} (i \neq j)$ by multiplication by λ_i and λ_j . On FE_{ij} , we have $L_d^* = \omega L_d - \omega^2 R_d = (\omega \lambda_i - \omega^2 \lambda_j) \text{Id}$ and $R_d^* = -\omega^2 L_d + \omega R_d = (-\omega^2 \lambda_i + \omega \lambda_j) \text{Id}$, so $L_d^* R_d^* = -(\lambda_i^2 + \lambda_j^2 + \lambda_i \lambda_j) \text{Id}$. Note that for $d = u, \lambda_1 = 2\rho, \lambda_2 = -\rho, \lambda_3 = -\rho$, this is $-((2\rho)^2 + (-\rho)^2 + (-2\rho^2))$ or $-((-\rho)^2 + (-\rho)^2 + (\rho)^2)$ both of which are $-3\rho^2 = 1$. Thus, $R_u^* = (L_u^*)^{-1}$.

Since $S(u) = \text{tr}(\text{adj}(u)) = \text{tr} \text{diag}\{\rho^2, -2\rho^2, -2\rho^2\} = 1$, we get $(F_3)_0$ with product $a \square b = (R_u^{*-1}a) * (L_u^{*-1}b) = (u * a) * (b * u)$ is an eight-dimensional unital composition algebra from [3]. Since $S(e_1) = \text{tr}(\text{adj}(e_1)) = \text{tr}(e_2) = 0$, it is a split octonion algebra.

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REFERENCES

1. B. N. Allison and J. R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, Trans. Amer. Math. Soc. **283** (1984), 185–210.
2. J. R. Faulkner, *A geometric construction of Moufang planes* (To appear in Geometrie Dedicata).
3. I. Kaplansky, *Infinite-dimensional quadratic forms admitting composition*, Proc. Amer. Math. Soc. **4** (1953), 956–960.
4. K. McCrimmon, *The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras*, Trans. Amer. Math. Soc. **139** (1969), 495–510.
5. ———, *Generically algebraic algebras*, Trans. Amer. Math. Soc. **127** (1967), 527–551.

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