SETS OF DISJOINT ELEMENTS IN FREE PRODUCTS OF LATTICE-ORDERED GROUPS

WAYNE B. POWELL AND CONSTANTINE TSINAKIS

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ABSTRACT. We show that for every infinite cardinal number m there exist two totally ordered abelian groups whose free product in any nontrivial variety of lattice-ordered groups has a disjoint set of cardinality m . This answers problem 10.7 of [13] and extends the results in [12]. We further prove that for the variety of abelian lattice-ordered groups or the variety of all lattice-ordered groups, the free product of two nontrivial members of the variety will always contain an infinite disjoint set.

1. Introduction. Let m be an infinite cardinal. An l -group (lattice-ordered group) G is said to satisfy the m -disjointness condition if for every $a \in G$ and every $S \subseteq G$ it follows that $|S| < m$ whenever $a \notin S$ and $x \wedge y = a$ for all distinct $x, y \in G$. Such a set S is said to be *disjoint* (with respect to a). Note that every group translation of an l -group is a lattice automorphism and hence one only need consider disjoint sets with respect to the group identity in deciding the failure or satisfaction of the m -disjointness condition.

The question of the preservation of the m -disjointness condition for free products in varieties of l -groups was addressed in [12]. It was shown there that if $\mathcal{A}$ is the variety of abelian l -groups, then the $\mathcal{A}$ -free product of two totally ordered members of $\mathcal{A}$ can have arbitrarily large cardinality. It was further shown that if $\mathcal{U}$ is any nontrivial variety of l -groups, then $\mathcal{U}$ -free products do not preserve the m -disjointness condition whenever m is regular but not weakly compact, countable, or singular. The proof of the latter result assumes the existence of an m -Souslin tree whenever m is a regular cardinal that is not weakly compact and is based on a related lattice theoretic result of Adams and Kelly [1]. Thus the following problem presents itself.

PROBLEM 1. Let $\mathcal{U}$ be any nontrivial variety of l -groups. Do $\mathcal{U}$ -free products fail to preserve the m -disjointness condition for all infinite cardinals m and, if so, can this failure be decided on the basis of the standard axioms of set theory?

The first objective of this paper is to provide an affirmative answer to this problem. More specifically, Theorem 4 asserts that given an infinite cardinal m , there exist totally ordered abelian groups G_1 and G_2 of cardinality m such that for any nontrivial variety $\mathcal{U}$ of l -groups the $\mathcal{U}$ -free product of G_1 and G_2 has a disjoint set of cardinality m . This result is actually an easy consequence of a more general

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result relating disjoint sets in free products with certain chains in the free factors (Theorem 3).

Recall that the lattice of an l -group is a distributive lattice (see Bigard et al. [3, p. 19]). Furthermore, there is a close connection between free products in varieties of l -groups and free products in the class of distributive lattices (Franchello [4], Powell and Tsinakis [11]). Thus the theorem above is somewhat surprising when contrasted with the fact that free products in the variety of distributive lattices preserve the m -disjointness condition whenever the cardinal m is singular or weakly compact (Adams and Kelly [1]).

A closely related problem is the following.

PROBLEM 2. Suppose that $\mathcal{U}$ is a nontrivial variety of l -groups. Does the $\mathcal{U}$ -free product of G_1 and G_2 have an infinite disjoint set whenever G_1 and G_2 are nontrivial members of $\mathcal{U}$?

It has been recently shown in Glass [6] that Problem 2 has an affirmative answer in the special case $\mathcal{U}$ is the variety of all l -groups and G_1 and G_2 are countable. We show that Problem 2 has an affirmative answer for the variety of abelian l -groups and that of all l -groups (Theorem 7) without any restrictions on G_1 and G_2 . A partial affirmative answer for an arbitrary $\mathcal{U}$ is provided by Theorem 8.

The aforementioned results would seem to indicate that, apart from trivial cases, free products never preserve disjointness conditions. However, this is not the case. For example, any $\mathcal{U}$ -free l -group of rank ≥ 2 satisfies the $\aleph_1$ -disjointness condition and fails the $\aleph_0$ -disjointness condition (see [12]). It follows that the $\mathcal{U}$ -free product of two such l -groups preserved the $\aleph_1$ -disjointness condition. It would be of interest to obtain general theorems along these lines.

Before we embark on the proof of the theorems, let us recall the concept of a free product and establish some notation.

Let $\mathcal{U}$ be a class of algebras of the same similarity type and let $(G_i | i \in I)$ be a family of members of $\mathcal{U}$. The $\mathcal{U}$ -free product of the family is an algebra $G \in \mathcal{U}$, denoted by $\mathcal{U} \bigsqcup_{i \in I} G_i$, together with a family of injective homomorphisms $(a_i: G_i \rightarrow G | i \in I)$ such that

- (i) $\bigcup_{i \in I} a_i(G_i)$ generates G ;
- (ii) if $H \in \mathcal{U}$ and $(\beta_i: G_i \rightarrow H | i \in I)$ is a family of homomorphisms, then there exists a (necessarily) unique homomorphism $\gamma: G \rightarrow H$ satisfying $\beta_i = \gamma a_i$ for all $i \in I$.

We will always identify each free factor G_i with its image $a_i(G_i)$ in $\mathcal{U} \bigsqcup_{i \in I} G_i$ and thus view each a_i as the inclusion map.

Given a family $(G_i | i \in I)$ in a variety $\mathcal{U}$, it is well known that $\mathcal{U} \bigsqcup_{i \in I} G_i$ exists if and only if there exists $H \in \mathcal{U}$ so that each G_i can be embedded in H . This condition is clearly satisfied when $\mathcal{U}$ is a variety of l -groups, for then one can let H be the (cardinal) product of the l -groups G_i . For information on properties of l -group free products it is best to first look at [13].

In what follows we use $\mathcal{U}$ to denote an arbitrary nontrivial variety of l -groups. Two varieties which have special significance and are given distinguished notation are the variety of abelian l -groups, denoted by $\mathcal{A}$, and the variety of all l -groups, denoted by $\mathcal{L}$. It is well-known that $\mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{L}$ (Weinberg [14]).

Group operations will be written multiplicatively throughout. The *cardinal sum* $\boxplus_{i \in I} H_i$ of a family $(H_i | i \in I)$ of l -groups is the group direct sum, $\bigoplus_{i \in I} H_i$, endowed

with the componentwise lattice order. If x is an element of an l -group, then its positive part, $x \vee 1$, is denoted by x^+ . The reader may consult Bigard et al. [3] and Glass [5] for elementary concepts not defined here.

2. Construction of the disjoint sets. If Ω is a chain, we denote by $A(\Omega)$ the l -group (under the pointwise order) of all order preserving permutations of Ω . An l -subgroup G of $A(\Omega)$ is referred to as an l -permutation group. When G is identified as a set of order preserving permutations from $A(\Omega)$, it is labelled (G, Ω) . Ω is said to be *doubly homogeneous* if $A(\Omega)$ is doubly transitive: whenever $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ are elements in Ω , there is a $g \in A(\Omega)$ such that $g(\alpha_1) = \beta_1$ and $g(\alpha_2) = \beta_2$. It is not hard to see that a doubly homogeneous chain is dense, that is, given $a_1 < a_2$ in Ω , there is $a_3 \in \Omega$ with $a_1 < a_3 < a_2$.

The following fundamental result due to Holland [7] (see Glass [5, p. 41 and p. 240]) will be used in some of the subsequent proofs.

LEMMA 1. *Every l -group can be embedded in $A(\Omega)$ for a suitable chain Ω . In fact, Ω can be chosen to be doubly homogeneous.*

LEMMA 2. *Let a_1, a_2, b_1, b_2 be positive elements of an l -group G such that $a_2^2 < a_1$ and $b_1^2 < b_2$. If*

$$x_i = [(a_i^2 b_i^{-1}) \wedge (b_i^2 a_i^{-1})]^+$$

for $i = 1, 2$, then $x_1 \wedge x_2 = 1$.

PROOF. In view of Lemma 1, we may view G as an l -permutation group (G, Ω) for a suitable chain Ω . Since $a_1^2 b_1^{-1} \geq a_2^2 b_2^{-1}$ and $b_2^2 a_2^{-1} \geq b_1^2 a_1^{-1}$, it follows that

$$1 \leq x_1 \wedge x_2 \leq (b_1^2 a_1^{-1})^+ \wedge (a_2^2 b_2^{-1})^+,$$

and hence it is sufficient to prove that the right-hand side of the inequality is equal to 1. This is of course equivalent to showing that for all $\omega \in \Omega$,

$$\min\{b_1^2 a_1^{-1}(\omega), a_2^2 b_2^{-1}(\omega)\} \leq \omega.$$

To this end, let $\omega \in \Omega$ and suppose that $b_1^2 a_1^{-1}(\omega) > \omega$, i.e., $a_1^{-1}(\omega) > b_1^{-2}(\omega)$. Now by hypothesis, $a_2^{-2}(\omega) \geq a_1^{-1}(\omega)$ and $b_1^{-2}(\omega) \geq b_2^{-1}(\omega)$. It follows that $a_2^{-2}(\omega) > b_2^{-1}(\omega)$ or $a_2^2 b_2^{-1}(\omega) < \omega$. $\square$

The next result shows that disjoint sets in a free product can be built from suitable chains in the free factors. Let G be an l -group and let m be a cardinal number. A chain $(a_\lambda | \lambda < m)$ in G of length m is said to be an *admissible ascending* (*descending*) chain if for all $\lambda < \mu < m$, $1 < a_\lambda^2 < a_\mu$ ($1 < a_\mu^2 < a_\lambda$).

THEOREM 3. *Let m be an infinite cardinal. Suppose that G_1 and G_2 are abelian l -groups such that G_1 has an admissible descending chain of length m and G_2 has an admissible ascending chain of length m . Then for every nontrivial variety $\mathcal{U}$ of l -groups, $G_1^{\mathcal{U}} \sqcup G_2$ has a disjoint set of cardinality m .*

PROOF. Let $(a_\lambda | \lambda < m)$ be an admissible descending chain in G_1 , and $(b_\lambda | \lambda < m)$ an admissible ascending chain in G_2 . Consider the elements

$$x_\lambda = [(a_\lambda^2 b_\lambda^{-1}) \wedge (b_\lambda^2 a_\lambda^{-1})]^+ \quad (\lambda < m)$$

in $G_1^{\mathcal{U}} \sqcup G_2$ and the set $S_{\mathcal{U}} = \{x_\lambda | \lambda < m\}$.

We have remarked that $\mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{L}$ and hence the identity mappings on G_1 and G_2 induce the epimorphisms

$$\phi_{\mathcal{U}}: G_1^{\mathcal{L}} \sqcup G_2 \rightarrow G_1^{\mathcal{U}} \sqcup G_2$$

and

$$\psi_{\mathcal{U}}: G_1^{\mathcal{U}} \sqcup G_2 \rightarrow G_1^{\mathcal{A}} \sqcup G_2.$$

The goal is to show that $S_{\mathcal{U}} = \{\phi_{\mathcal{U}}(x_{\lambda}) \mid \lambda < m\}$ is a disjoint set in $G_1^{\mathcal{U}} \sqcup G_2$ of cardinality m . The proof below incorporates ideas from the proof of Theorem 2.1 of [12].

To begin with, in view of Lemma 2, it is sufficient to prove that $S_{\mathcal{U}} \cap \{1\} = \emptyset$. Further, since every $S_{\mathcal{U}}$ is projected onto $S_{\mathcal{A}}$ by $\psi_{\mathcal{U}}$, we only need show that $S_{\mathcal{A}} \cap \{1\} = \emptyset$. To this end, we fix $\lambda < m$ and proceed to show that $\phi_{\mathcal{A}}(x_{\lambda}) \neq 1$ in $G_1^{\mathcal{A}} \sqcup G_2$. Let P_i be a prime subgroup of G_i ($i = 1, 2$) such that $a_{\lambda} \notin P_1$ and $b_{\lambda} \notin P_2$. Consider the totally ordered groups $H_i = G_i/P_i$ ($i = 1, 2$) and let $\psi: G_1^{\mathcal{A}} \sqcup G_2 \rightarrow H_1^{\mathcal{A}} \sqcup H_2$ be the epimorphism extending the natural projections. It will be sufficient to verify that $\psi\phi_{\mathcal{A}}(x_{\lambda}) \neq 1$ in $H_1^{\mathcal{A}} \sqcup H_2$. Set $\psi(a_{\lambda}) = a$ and $\psi(b_{\lambda}) = b$. In view of the universal property of $\mathcal{A}$ -free products, it will be enough to prove that the elements a^2b^{-1} and b^2a^{-1} in $H_1 \oplus H_2$ are positive with respect to some total order on $H_1 \oplus H_2$ extending the orders of H_1 and H_2 . Now any partial order on a torsion-free abelian group can be extended to a total order (see for example Mura and Rhemtulla [9, §3.1]), and thus it will suffice to prove that there is a partial order on $H_1 \oplus H_2$ whose cone contains the set $(H_1 \boxplus H_2)^+ \cup \{a^2b^{-1}, b^2a^{-1}\}$. It is easy to see that such a partial order exists if and only if for all $n_1, n_2 \in \mathbb{Z}^+ \setminus \{0\}$, the combination $(a^2b^{-1})^{n_1}(a^2b^{-1})^{n_2}$ is not negative in $H_1 \boxplus H_2$. Thus suppose that $(a^2b^{-1})^{n_1}(a^{-1}b^2)^{n_2} \leq 1$ in $H_1 \boxplus H_2$ for some $n_1, n_2 \in \mathbb{Z}^+$. Then $a^{2n_1-n_2} \leq 1$ and $b^{2n_2-n_1} \leq 1$ in H_1 and H_2 , respectively. As $a > 1$ in H_1 and $b > 1$ in H_2 , this implies $2n_1 - n_2 \leq 0$ and $2n_2 - n_1 \leq 0$ and so $n_1 = n_2 = 0$. $\square$

We can now use Theorem 3 to give an affirmative answer to Problem 1.

THEOREM 4. *Let m be an infinite cardinal. There exists totally ordered abelian groups G_1 and G_2 of cardinality m such that for any nontrivial variety $\mathcal{U}$ of l -groups, $G_1^{\mathcal{U}} \sqcup G_2$ has a disjoint set of cardinality m .*

PROOF. Let Λ be a totally ordered set of cardinality m . Define

$$G_1 = \bigoplus_{\lambda \in \Lambda} G_{\lambda}^{(1)} \quad \text{and} \quad G_2 = \bigoplus_{\lambda \in \Lambda} G_{\lambda}^{(2)},$$

where $G_1 \cap G_2 = \{1\}$, and for each $\lambda \in \Lambda$, $G_{\lambda}^{(1)}$ and $G_{\lambda}^{(2)}$ are isomorphic to the totally ordered group $\mathbb{Z}$ of integers. Endow G_1 and G_2 with the lexicographic and the dual lexicographic order respectively. Thus if $x = x_1x_2 \cdots x_n$ with $x_i \in G_{\lambda_i}^{(1)} \setminus \{1\}$ (respectively, $x_i \in G_{\lambda_i}^{(2)} \setminus \{1\}$) and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then $x > 1$ in G_1 iff $x_1 > 1$ in $G_{\lambda_1}^{(1)}$ (and $x > 1$ in G_2 iff $x_n > 1$ in $G_{\lambda_n}^{(2)}$).

Now for each $\lambda \in \Lambda$, pick $a_{\lambda} \in (G_{\lambda}^{(1)})^+ \setminus \{1\}$ and $b_{\lambda} \in (G_{\lambda}^{(2)})^+ \setminus \{1\}$. It is readily seen that $(a_{\lambda} \mid \lambda \in \Lambda)$ is an admissible descending chain in G_1 and $(b_{\lambda} \mid \lambda \in \Lambda)$ is an admissible ascending chain in G_2 . The result now follows from Theorem 3 since these two chains have length m . $\square$

It would be of interest to determine whether Theorem 3 remains true when G_1 and G_2 are arbitrary l -groups in $\mathcal{U}$. The following result shows that this is the case if $\mathcal{U} = \mathcal{L}$.

THEOREM 5. *Let m be an infinite cardinal and let $\mathcal{U} = \mathcal{A}$ or $\mathcal{L}$. Suppose that G_1 and G_2 are l -groups in $\mathcal{U}$ such that G_1 has an admissible descending chain of length m and G_2 has an admissible chain of length m . Then $G_1^{\mathcal{U}} \sqcup G_2$ has a disjoint set of cardinality m .*

PROOF. The case $\mathcal{U} = \mathcal{A}$ is found in Theorem 3. Thus suppose that $\mathcal{U} = \mathcal{L}$. Let $(a_\lambda | \lambda < m)$ be an admissible descending chain in G_1 and let $(b_\lambda | \lambda < m)$ be an admissible ascending chain in G_2 . Define the subset $S_{\mathcal{L}} = \{x_\lambda | \lambda < m\}$ of $G = G_1^{\mathcal{L}} \sqcup G_2$ as the proof of Theorem 3:

$$x_\lambda = [(a_\lambda^2 b_\lambda^{-1}) \wedge (b_\lambda^2 a_\lambda^{-1})]^+.$$

In view of Lemma 2, we only need show that $S_{\mathcal{L}} \cap \{1\} = \emptyset$. To this end, we fix $\lambda < m$ and proceed to show that $x_\lambda \neq 1$. By Lemma 1, we may view G as an l -permutation group (G, Ω) , where $A(\Omega)$ is doubly transitive. Now $a_\lambda, b_\lambda > 1$ and hence there exist $\omega_1, \omega_2 \in \Omega$ such that $a_\lambda^{-1}(\omega_1) < \omega_1$ and $b_\lambda^{-1}(\omega_2) < \omega_2$. Thus, since Ω is doubly homogeneous, there exists $h \in A(\Omega)$ such that $h(\omega_2) = \omega_1$ and $h(b_\lambda^{-1}(\omega_2)) = a_\lambda^{-1}(\omega_1)$. We have $a_\lambda^{-1}(\omega_1) = c_\lambda^{-1}(\omega_1) < \omega_1$, where $c_\lambda = hb_\lambda h^{-1}$. It follows that

$$a_\lambda^2 c_\lambda^{-1}(\omega_1) = a_\lambda(\omega_1) > \omega_1$$

and

$$c_\lambda^2 a_\lambda^{-1}(\omega_1) = c_\lambda(\omega_1) > \omega_1.$$

This shows that

$$(*) \quad (a_\lambda^2 c_\lambda^{-1})^+ \wedge (c_\lambda^2 a_\lambda^{-1})^+ > 1$$

in $A(\Omega)$. Consider now the l -homomorphism $f: G \rightarrow A(\Omega)$ extending the inclusion map $f_1: G_1 \rightarrow A(\Omega)$ and the l -homomorphism $f_2: G_2 \rightarrow A(\Omega)$ defined by $f_2(g) = hgh^{-1}$ for all $g \in G_2$. In view of $(*)$, $f(x_\lambda) > 1$ in $f(G)$ and hence $x_\lambda > 1$ in G . This completes the proof of the theorem. $\square$

The remainder of the paper is concerned with Problem 2. Given an l -group G , we write G' for the l -ideal of G generated by the derived subgroup.

THEOREM 6. *Let $\mathcal{U}$ be a nontrivial l -group variety and let G_1, G_2 be l -groups in $\mathcal{U}$ such that $G'_1 \neq G_1$ and $G'_2 \neq G_2$. Then $G_1^{\mathcal{U}} \sqcup G_2$ has an infinite disjoint set.*

PROOF. Let $x \in G_1^+ \setminus G'_1$ and $y \in G_2^+ \setminus G'_2$. For each $n \in \mathbf{Z}^+ \setminus \{0\}$, let

$$s_n = [(x^{-(n-1)}y) \wedge (y^{-1}x^n)]^+$$

in $G = G_1^{\mathcal{U}} \sqcup G_2$. We claim that

$$S = \{s_n | n \in \mathbf{Z}^+ \setminus \{0\}\}$$

is an infinite disjoint set in G .

To begin with, let $m, n \in \mathbf{Z}^+ \setminus \{0\}$ with $m < n$. Then

$$1 \leq s_m \wedge s_n \leq (y^{-1}x^m)^+ \wedge (x^{-m}y)^+ = 1.$$

We complete the proof by showing that $s_n \neq 1$ for all n . Fix $n \in \mathbf{Z}^+ \setminus \{0\}$. Note that G/G' is a nontrivial abelian l -group whenever $G' \neq G$. Furthermore, each abelian l -group is a subdirect product of totally ordered groups. Hence, for $i = 1, 2$, there exists an l -ideal P_i of G_i such that $G_i/P_i = H_i$ is a totally ordered abelian group, $x \notin P_1$, and $y \notin P_2$. As $\mathcal{A} \subseteq \mathcal{U}$, the natural projections $G_1 \rightarrow H_1$ and $G_2 \rightarrow H_2$ induce an epimorphism $p: G_1^{\mathcal{U}} \sqcup G_2 \rightarrow H_1^{\mathcal{A}} \sqcup H_2$. Thus it will suffice to show that $p(s_n) \neq 1$. This in turn is equivalent to showing that for all $n_1, n_2 \in \mathbf{Z}^+ \setminus \{0\}$, the element $(p(x)^{-(n-1)}p(y))^{n_1}(p(y)^{-1}p(x)^n)^{n_2}$ is not negative in $H_1 \boxplus H_2$. (See the proof of Theorem 3.) Thus suppose that $(p(x)^{-(n-1)}p(y))^{n_1}(p(y)^{-1}p(x)^n)^{n_2} \leq 1$ in $H_1 \boxplus H_2$ for some $n_1, n_2 \in \mathbf{Z}^+$. Then

$$p(x)^{nn_2-(n-1)n_1} \leq 1$$

and

$$p(y)^{n_1-n_2} \leq 1$$

in H_1 and H_2 respectively. As $p(x) > 1$ in H_1 and $p(y) > 1$ in H_2 , it follows that $n(n_2 - n_1) + n_1 \leq 0$ and $n_2 - n_2 \geq 0$. Thus $n_1 = n_2 = 0$, as was to be shown. $\square$

We use Theorem 6 to provide an affirmative answer to Problem 2 when $\mathcal{U} = \mathcal{A}$ or $\mathcal{L}$.

THEOREM 7. *If $\mathcal{U} = \mathcal{A}$ or $\mathcal{L}$, then the $\mathcal{U}$ -free product of two nontrivial l -groups in $\mathcal{U}$ has an infinite disjoint set.*

PROOF. The case $\mathcal{U} = \mathcal{A}$ follows directly from Theorem 6. Suppose that $\mathcal{U} = \mathcal{L}$ and let G_1, G_2 be two nontrivial l -groups in $\mathcal{L}$. Pick $x \in G_1^+ \setminus \{1\}$ and $y \in G_2^+ \setminus \{1\}$. For each $n \in \mathbf{Z}^+ \setminus \{0\}$, let

$$s_n = [(x^{-(n-1)}y) \wedge (y^{-1}x^n)]^+$$

in $G = G_1^{\mathcal{L}} \sqcup G_2$. We proceed to prove that

$$S = \{s_n | n \in \mathbf{Z}^+ \setminus \{0\}\}$$

is an infinite disjoint set in G . It was observed in the proof of Theorem 6 that if $m \neq n$, then $s_m \wedge s_n = 1$. Thus it will suffice to show that $s_n \neq 1$ for all n . Fix $n \in \mathbf{Z}^+ \setminus \{0\}$ and view G as an l -permutation group (G, Ω) , where Ω is doubly homogeneous. Note that $x^n > x^{n-1}$ and $y > 1$. Hence there are elements $\omega, \omega_1 \in \Omega$ such that $x^n(\omega) > x^{n-1}(\omega)$ and $y(\omega_1) > \omega_1$. Choose $\omega_2 \in A(\Omega)$ with $x^n(\omega) > \omega_2 > x^{n-1}(\omega)$ and let $h \in A(\Omega)$ such that $h(\omega_1) = \omega$ and $h(y(\omega_1)) = \omega_2$. Thus

$$x^n(\omega) > z(\omega) > x^{n-1}(\omega),$$

where $z = hyh^{-1}$. It follows that

$$(*) \quad [(x^{-(n-1)}z) \wedge (z^{-1}x^n)]^+ > 1$$

in $A(\Omega)$. Consider the l -homomorphism $f: G \rightarrow A(\Omega)$ extending the inclusion map $f_1: G_1 \rightarrow G(\Omega)$ and the l -homomorphism $f_2: G_2 \rightarrow (\Omega)$ defined by $f_2(g) = hgh^{-1}$, for all $g \in G_2$. In view of $(*)$, $f(s_n) > 1$ in $f(G)$ and hence $s_n > 1$ in G . This completes the proof of the theorem. $\square$

The techniques of the present paper are not sufficiently general to settle Problem 2. However, the following partial result can be easily established using Theorem 6.

THEOREM 8. *If $\mathcal{U}$ is a proper l -group variety and G_1, G_2 are nontrivial l -groups in $\mathcal{U}$ with a strong order unit, then $G_1^{\mathcal{U}} \sqcup G_2$ has an infinite disjoint set.*

PROOF. Every proper variety of l -groups is contained in the variety of normal valued l -groups (Holland [7]). Hence it is easy to verify that if G_1 and G_2 satisfy the conditions stated above, then $G_1 \neq G'_1$ and $G_2 \neq G'_2$. The conclusion now follows from Theorem 6.

We finally mention that Theorem 6 implies that for every nontrivial l -group variety $\mathcal{U}$, any $\mathcal{U}$ -free l -group of rank ≥ 2 contains an infinite disjoint set. This result was first established in [12]. That such an l -group satisfies the $\aleph_1$ -disjointness condition was proved in [2 and 10].

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37235