ON THE MONODROMY GROUP OF EVERYWHERE TANGENT LINES TO THE OCTIC SURFACE IN $\mathbf{P}^3$

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ABSTRACT. Let S_0 be an octic surface in $\mathbf{P}^3$, $G = G(1, 3) =$ Grassmannian of lines in $\mathbf{P}^3$, and $\mathbf{J} = \{(x, l) | x \in l \cap S_0\} \subset S_0 \times G$. Then $\dim \mathbf{J} = 5$. Let $\mathbf{L} = \{l | l \text{ is everywhere tangent to } S_0\}^- \subset G$. Let $\pi_2: S_0 \times G \rightarrow G$ be the projection onto the second factor. We denote its restriction to $\mathbf{J}$ also by π_2 . Then the locus of everywhere tangent lines is $\pi_2(\mathbf{L})$. In this article we show that the monodromy group of these lines is the full symmetric group.

Introduction. In this article we show that a general octic surface in $\mathbf{P}^3$, has a finite number of everywhere tangent lines and using the Harris technique in [H], we study the monodromy group of these lines and prove that this group is the full symmetric group.

Notation and background material.

(0.1) Let S_0 be an octic surface in $\mathbf{P}^3$. First we note that if S_0 is general then it cannot contain lines. Let $G = G(1, 3) =$ Grassmannian of lines in $\mathbf{P}^3$. Let $\mathbf{J} = \{(x, l) | x \in l \cap S_0\} \subset S_0 \times G$. We also note that $\dim \mathbf{J} = 2 + 4 - 1 = 5$.

(0.2) **DEFINITION.** A line l in $\mathbf{P}^3$ is said to be everywhere tangent to an octic surface S in $\mathbf{P}^3$, if the multiplicity at each of the points of intersection of S and l in $\mathbf{P}^3$ is at least two.

(0.3) Let $\mathbf{L} = \{(x, l) | l \text{ is everywhere tangent to } S_0\}^- \subset \mathbf{J}$; also let $\pi_2: S_0 \times G \rightarrow G$ be the projection onto the second factor. We denote its restriction to $\mathbf{J}$ also by π_2 , then the locus of everywhere tangent lines to S_0 is $\pi_2(\mathbf{L})$. From now on we denote $\pi_2(\mathbf{L})$ by $\mathbf{F}$.

(0.4) Let X be a smooth complex projective variety of dimension n .

Let $\text{Hilb}_c^r X$ denote the curvilinear Hilbert scheme of r points of X . Then $\text{Hilb}_c^r X$ is smooth of dimension nr (see [LB] for details).

If Y is a smooth hypersurface in X , then there is a natural inclusion $\text{Hilb}_c^r Y \rightarrow \text{Hilb}_c^r X$ of dimension $(n - 1)r$.

Let $\text{Hilb}_c^r \underline{S}/\mathbf{P}^N$ denote the relative (curvilinear) Hilbert scheme of the universal family of octic surfaces parametrized by $\mathbf{P}^N$; we note that $N = 164$.

The linear system of all octics in $\mathbf{P}^3$ is denoted by W_8 , and is of dimension $N = 164$. If S denotes a general octic surface in $\mathbf{P}^3$ (see (0.6)), then the relative (curvilinear) Hilbert scheme above is of dimension $2r + N$.

(0.5) Let $\mathbf{a} = (a_1, \dots, a_k)$ where $a_i \leq a_j$ where $i < j$ and $a_i \geq 2$, and $r = \sum a_i = 8$. We note that $\mathbf{a} = (2, 2, 2, 2); (2, 2, 4); (2, 3, 3); (2, 6); (3, 5); (4, 4); (8)$, are the only possibilities.

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In what follows, the notation is borrowed from [Co], where one can check for details.

Let $\mathbf{a} = (b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_p, \dots, b_p)$ where $b_i < b_j$ for $i < j$, and $k = \sum q_i$ where q_i denotes the number of b_i , for $1 \leq i \leq p$. Consider $j_{\mathbf{a}}: \text{Hilb}^{q_1} \mathbf{P}^1 \times \dots \times \text{Hilb}^{q_p} \mathbf{P}^1 \rightarrow \text{Hilb}^r \mathbf{P}^1 \cong \mathbf{P}^r$ given by

$$(I_1, \dots, I_p) \mapsto I_1^{b_1}, \dots, I_p^{b_p}$$

where I_j is the ideal of the zero-dimensional length q_j subscheme of $\mathbf{P}^1$. We note that $j_{\mathbf{a}}$ is an embedding. Using $j_{\mathbf{a}}$ we can construct a flat family over G , denoted by $D_{\mathbf{a}}$, whose fibres are all isomorphic to the image of $j_{\mathbf{a}}$, and $D_{\mathbf{a}}$ embeds naturally in the curvilinear Hilbert scheme $\text{Hilb}_c^r \mathbf{P}^3$, and $\dim D_{\mathbf{a}} = k + 4$.

(0.6) DEFINITION. Consider $P_{\mathbf{a}} = D_{\mathbf{a}} \cap \text{Hilb}_c^r S$, where S is an octic surface in $\mathbf{P}^3$. S is said to be *general* if $P_{\mathbf{a}}$ is empty or $P_{\mathbf{a}}$ has the expected dimension $(k + 4) + 2r - 3r = k - 4$ for all possible partitions of 8 ($= \deg S$).

General results.

(1.0) PROPOSITION. *There exists a nonempty open subset U of $W_8 (\cong \mathbf{P}^N) N = 164$, (see (0.4)) such that any member of U is general in the sense of (0.6).*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc} X & \rightarrow & \text{Hilb}_c^r \underline{S}/\mathbf{P}^N \rightarrow \mathbf{P}^N \\ \downarrow & & \downarrow \\ D_{\mathbf{a}} & \rightarrow & \text{Hilb}_c^r \mathbf{P}^3 \\ \downarrow & & \\ G & & \end{array}$$

where $X = D_{\mathbf{a}} \times_{\text{Hilb}_c^r \mathbf{P}^3} \text{Hilb}_c^r \underline{S}/\mathbf{P}^N$, is the fiber product. The fiber of X over l ($\in G$) consists of all type- $\mathbf{a}$ subschemes of l which also lies on some octic surface S . Hence the dimension of the fiber is $(N - r) + k = 164 - 8 + k = 156 + k$. Also both l and S lie in some $\mathbf{P}^3$. Hence $\dim X = 160 + k$. Hence the general fiber over $\mathbf{P}^N$ is of dimension $160 + k - N = k - 4$. But the fiber of a general $x \in \mathbf{P}^N$ is just $P_{\mathbf{a}}$ (see (0.6)). Hence for an open subset U of $\mathbf{P}^N$, $P_{\mathbf{a}}$ is as desired. $\square$

(1.1) COROLLARY. *If S_0 is a general octic in $\mathbf{P}^3$ then $\mathbf{F}$ is a finite set. Moreover only the $(2, 2, 2, 2)$ type (see (0.5)) can possibly occur.*

PROOF. That $\mathbf{F}$ is finite is immediate from (1.0). Since S_0 is of degree 8, we have $l.S_0 = 8$ in $\mathbf{P}^3$. Let $\mathbf{a} = (a_1, \dots, a_k)$ where $a_i \geq 2$ and $a_i \leq a_j$ for $i < j$. Let $r = \sum a_i = 8$, where a_i denotes the multiplicity of l at the point of intersection with S_0 , then it is easy to see that the set of l with $\mathbf{a} = (2, 2, 4); (2, 3, 3); (2, 6); (3, 5); (4, 4)$ or (8) has $k < 4$, (see (0.5)) in each case and so for the open set U in (1.0) $P_{\mathbf{a}}$ is empty. In the case $(2, 2, 2, 2); k = 4$, and so this can occur. $\square$

Main results. First we show that $\mathbf{F}$ is not empty, by showing the existence of a general octic surface with an everywhere tangent line. We need the following lemmas and this remark.

(2.0) REMARK. In the following we let w_8 = the linear system of all octics in $\mathbf{P}^2$, and W_8 = the linear system of all octic surfaces in $\mathbf{P}^3$.

(2.1) LEMMA. *Given a line l in $\mathbf{P}^2$ and 4 distinct points on it, there exists an octic curve, C in $\mathbf{P}^2$ such that $m_p(l.C) = 2$, i.e. the line intersects each of these points with multiplicity 2.*

PROOF. If $G = G(1, 2) = \text{Grassmannian of all lines in } \mathbf{P}^2$, then $\dim G = 2$ and $\dim w_8 = 44$. Let p_1, p_2, p_3 and p_4 denote the 4 specified points on l .

Let w' be the space of octics having the tangent line l at these 4 points p_i , then $\dim w' = \dim w - [2 \text{ for each } p_i] = 44 - 8 = 36$. Also the space of all marked lines in $\mathbf{P}^2$ passing through 4 marked points p_i is of dimension, $\dim \mathbf{P}^2 + [1 \text{ for each } p_i] = 2 + 4 = 6$.

Let w_0 be the space of octics with everywhere tangent lines in $\mathbf{P}^2$; then $\dim w_0 = 36 + 6 = 42$. $\square$

(2.2) LEMMA. *Given a line l in $\mathbf{P}^2$ and 3 distinct points say p', p'' and p''' on it, there exists an octic curve, C in $\mathbf{P}^2$ such that $m_{p'}(l.C) = 2 = m_{p''}(l.C)$ and $m_{p'''}(l.C) = 4$.*

PROOF. Let w' be as in (2.1). Then as in (2.1) we see that $\dim w' = \dim w - [2 \text{ for each of } p' \text{ and } p'' \text{ and } 4 \text{ for } p'''] = 44 - 8 = 36$ and the space of all marked lines in $\mathbf{P}^2$ passing through 3 marked points p', p'' and p''' is of dimension, $\dim \mathbf{P}^2 + [1 \text{ for each of } p', p'' \text{ and } p'''] = 2 + 3 = 5$. Thus the dimension of the space of such octics is $36 + 5 = 41$. $\square$

(2.2.1) Let w_1 be the space of octics satisfying (2.2) then by (2.2), $\dim w_1 = 41$.

(2.3) LEMMA. *For a general octic surface in $\mathbf{P}^3$, there are only finitely many l with $\mathbf{a} = (2, 2, 2, 2)$. Moreover the set of such l is nonempty.*

PROOF. Consider $(\mathbf{P}^3)^*$, and let ζ be the tautological bundle on $(\mathbf{P}^3)^*$. This bundle is nontrivial. Choose a Zariski open subset, say A , of $(\mathbf{P}^3)^*$ over which this bundle is trivial.

For $H \in A$, let $w_S = \{C_H | C_H = H \cap S, S \text{ a fixed general octic in } \mathbf{P}^3\} \subset w_8$ (see (2.0)), then via the map $\Psi: A \rightarrow w_8$ given by $H \mapsto H \cap S$ we see that $\dim w_S = 3$. Hence if $\mathbf{a} = (2, 2, 2, 2)$, by (2.1), $\dim(w_0 \cap w_S) \geq \dim w_0 + \dim w_S - \dim w_8 = 42 + 3 - 44 = 1$. Hence $\mathbf{F}$ is (1.0) is a nonempty finite set. $\square$

(2.4) LEMMA. *Using the notations of (1.1) there exists an octic surface in $\mathbf{P}^3$, and a tangent line l with $\mathbf{a} = (0, 2, 2, 4)$.*

PROOF. We note that by (1.1), S_1 cannot be general. Let $\mathbf{a} = (0, 2, 2, 4)$ and w_1 as in (2.2.1); suppose $C_1 \in w_1$. Let S_1 be an octic surface in $\mathbf{P}^3$ containing C_1 , such a surface S_1 must exist by (2.2). $\square$

(2.5) Let W_8 be as in (2.0). We note that $\dim W_8 = 164$. Let S be a general octic in $\mathbf{P}^3$; let $m_p(S.l)$ denote the intersection multiplicity of S and l , in $\mathbf{P}^3$, and let $I_8 = \{(S; p, l) | l \text{ is everywhere tangent to } S, \text{ and } p \in l \cap S\} \subset W_8 \times \mathbf{P}^3 \times G$. Let $\pi: I_8 \rightarrow W_8$ and $\eta: I_8 \rightarrow \mathbf{P}^3 \times G$ be the projections onto the first and the last two factors respectively.

(2.6). REMARKS. (i) By (1.1), (2.1) and (2.2), π is generically finite.

(iii) Since the fibers of η are linear spaces of dimension 157, I_8 is irreducible.

(2.7) LEMMA. *The monodromy group M , of π is twice transitive.*

PROOF. First we show that M is transitive. Let U be a Zariski open subset of W_8 over which π is unramified, then $V = \pi^{-1}(U)$ is irreducible. Let $S \in U$,

and let $(S; p', l')$ and $(S; p'', l'') \in \pi^{-1}(S)$ with $m_{p'}(S.l') = 2$ and $m_{p''}(S.l'') = 2$, respectively. Draw an arc $\gamma^\sim$ on V with $\gamma^\sim(0) = (S; p', l')$ and $\gamma^\sim(1) = (S; p'', l'')$; then the monodromy action associated to the arc $\gamma = \pi \circ \gamma^\sim$ will carry $(S; p', l')$ to $(S; p'', l'')$. Thus M is transitive.

Now let S_0 be a *general* octic with $p_0 \in S_0$, and (p_0, l_0) such that $m_p(S_0.l_0) = 2$ and $(S_0; p_0, l_0) \in I_8$, i.e. l_0 is also everywhere else tangent to S_0 . Let $W_0 = \pi(\eta^{-1}(p_0, l_0)) = \{S | m_p(S.l_0) \geq 2\} \leq W_8$; we note that $S_0 \in W_0$ and let $I = \{(S; p, l) | m_p(S.l) \geq 2, p \neq p_0, l \neq l_0\} \subset I_8$, then η maps I onto a Zariski open subset $\{(p, l) | p \neq p_0, l \neq l_0\}$ of $\mathbf{P}^3 \times G$, and as in (2.6), I is irreducible. Hence every Zariski open subset of I is connected. Suppose $(S_0; p', l')$, $(S_0; p'', l'')$ are two other points lying in $\pi^{-1}(S_0)$, then we can find an arc $\gamma^\sim$ in I such that $\gamma^\sim(0) = (S_0; p', l')$ and $\gamma^\sim(1) = (S_0; p'', l'')$. Note that $\pi^{-1}(S_0)$ has all points *except* $(S_0; p_0, l_0) \in I$; then the monodromy action associated to the arc $\gamma = \pi \circ \gamma^\sim$ carries $(S_0; p', l')$ to $(S_0; p'', l'')$ and leaves $(S_0; p_0, l_0)$ fixed. Hence the stabilizer of $(S_0; p_0, l_0)$ in M acts transitively on the remaining points. Hence M is twice transitive. $\square$

(2.8) LEMMA. *Let M be as in (2.7), then M contains a simple transposition.*

PROOF. By (2.4) we know that there exists an octic surface S_1 in W_8 , with a line l , such that $m_{p'}(S_1.l) = 2 = m_{p''}(S_1.l)$ and $m_{p'''}(S_1.l) = 4$. Let U be a Zariski open subset of W_8 , over which π is unramified. Let $V = \pi^{-1}(U)$, and let Δ be an open neighbourhood of S_1 in W_8 , so chosen, so that the points $(S_1; p', l)$, $(S_1; p'', l)$ and $(S_1; p''', l)$ have disjoint neighbourhoods Δ' , Δ'' and Δ''' respectively, each mapping onto Δ , via π . Since Δ''' is irreducible, $\Delta''' \cap V$ is connected. Since I_8 is irreducible, it is locally irreducible. Thinking of $(S_1; p''', l)$ as two infinitely near points q and r each with multiplicity two i.e. $m_q(S_1, l) = 2 = m_r(S_1, l)$, we can find $\gamma^\sim$ in $\Delta''' \cap V$ such that $\gamma^\sim(0) = (S_1; q, l)$ and $\gamma^\sim(1) = (S_1; r, l)$; hence $\gamma = \pi \circ \gamma^\sim$ will fix $(S_1; p', l)$ and $(S_1; p'', l)$ and γ interchanges q and r . Hence M contains a simple transposition. $\square$

(2.9) THEOREM. *The monodromy group M , of everywhere tangent lines to a general octic surface in $\mathbf{P}^3$, is the full symmetric group.*

PROOF. By (2.7) and (2.8), for a general octic surface, the monodromy group M , is twice transitive and contains a simple transposition; hence it contains all simple transpositions. So M is necessarily the entire symmetric group. $\square$

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