A SYMMETRIC DENSITY PROPERTY: MONOTONICITY AND THE APPROXIMATE SYMMETRIC DERIVATIVE

C. FREILING AND D. RINNE

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ABSTRACT. The following is established:

Let W and B be open sets of real numbers whose union has full measure. If for each x , the set $\{h > 0 | x - h \in W, x + h \in B\}$ has density zero at zero, then these sets are all empty.

This is then used to prove the following:

If f is a continuous real valued function with a nonnegative lower approximate symmetric derivative, then f is nondecreasing.

Introduction. We will establish the following result.

Let W and B be open subsets of an interval whose union has full measure in this interval. If for each x in the interval, the set $\{h > 0 | x - h \in W, x + h \in B\}$ has density zero at zero, then these sets are all empty.

We then use this to prove the following.

If f is a continuous real valued function with a nonnegative lower approximate symmetric derivative, then f is nondecreasing.

Early attempts to prove this theorem can be found in [1] and [5]. More recently, some partial results have been established [3, 4, 6]. A survey of this and related topics appears in Larson [2].

DEFINITIONS. We will use the following notation and definitions. All functions are real valued and λ is Lebesgue measure on the real line. For a set A , $\chi_A(x)$ is the characteristic function of A . For A measurable, the upper right density of A at x is $\bar{d}^+(A, x) = \limsup \lambda(A \cap (x, x + h))/h$ as $h \rightarrow 0^+$. The upper left, lower right, and lower left densities, $\bar{d}^-$, $\underline{d}^+$, and $\underline{d}^-$ are defined in a similar manner. If these four values are the same, their common value is $d(A, x)$, the density of A at x . The upper right Dini derivate of f is $D^+f(x) = \limsup (f(x + h) - f(x))/h$ as $h \rightarrow 0^+$. The upper left, lower right, and lower left derivates, D^-f , D_+f , and D_-f are defined similarly. For sets A and B , we define $AB(x) = \{h > 0 | x - h \in A, x + h \in B\}$. The lower approximate symmetric derivative of f , $\underline{f}_{ap}^{(1)}(x)$, is the least upper bound of the set of α such that $\{h | (f(x + h) - f(x - h))/2h < \alpha\}$ has density zero at 0. The upper approximate symmetric derivative, $\bar{f}_{ap}^{(1)}(x)$, is defined similarly. If $\underline{f}_{ap}^{(1)}(x) = \bar{f}_{ap}^{(1)}(x)$, their common value is the approximate symmetric derivative of f , $f_{ap}^{(1)}(x)$.

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A symmetric density property for open sets.

LEMMA 1. Let W be a measurable subset of (a, b) with $\bar{d}^+(W, x) \geq \bar{d}^-(W, x)$ and $\underline{d}^+(W, x) \geq \underline{d}^-(W, x)$ for all x in (a, b) . If $d(W, z) = 1$ and $d(W, y) = 0$ for some $z < y$ and $0 < \gamma < 1$, then $d(W, x) = \gamma$ for some x between z and y .

PROOF. Let $f(x) = \int_a^x \chi_w(t) dt$. Then f is continuous on $[a, b]$ and $D^+f \geq D^-f$ and $D_+f \geq D_-f$ on (a, b) . Suppose $f'(z) = 1$ and $f'(y) = 0$ for some $z < y$ and $0 < \gamma < 1$. Let $g(x) = f(x) - \gamma x$. Then $D^+g \geq D^-g$ and $D_+g \geq D_-g$ on (a, b) . Since $g'(z) = 1 - \gamma > 0$ and $g'(y) = -\gamma < 0$, g assumes its maximum on $[z, y]$ at some x in (z, y) . Since $D^+g(x) \leq 0$ and $D^-g(x) \geq 0$, we have $D^+g(x) = D^-g(x) = 0$. Also, $D_-g(x) = 0$. Finally, $D_+g(x) \leq 0$, so $D_+g(x) = 0$. Thus $g'(x) = 0$ and $f'(x) = \gamma$, so $d(W, x) = \gamma$ as required. This finishes the proof of the lemma.

THEOREM 1. Let W and B be open subsets of $[a, b]$ with $\lambda(W \cup B) = b - a$. If $WB(x)$ has density 0 at 0 for each x in (a, b) , then $WB(x) = \emptyset$ for each x in (a, b) .

PROOF. Suppose $WB(z) \neq \emptyset$ for some z and $WB(x)$ has density 0 at 0 for all x in (a, b) . It is easy to see that the density requirement forces $W \cap B$ to be empty. Adjoin to W any points contained in open intervals in which W has full measure. Do the same for B . Clearly, W and B are still open and disjoint, and the density of $WB(x)$ has not been altered. Let $Q = [a, b] - (W \cup B)$, a closed set, and define C to be the set of points in Q that are both right endpoints of components of B and left endpoints of components of W . By Lemma 1, if $x \in W$ and $y \in B$ with $x < y$, then $[x, y] \cap Q$ is uncountable, since the density of W can be strictly between 0 and 1 only at points in Q . This says that Q is the disjoint union $P' \cup C$, where P' is a perfect set. Let $P_n = \{x \in P' \mid \lambda(WB(x) \cap (0, \delta)) / \delta \leq 0.0001 \text{ for all } 0 < \delta \leq 1/n\}$. It is easy to see that each P_n is closed. Since $P' = \bigcup_{n=1}^{\infty} P_n$, we can use the Baire Category Theorem to get a subinterval of P' contained in some P_n . Let $P'' = [c, d] \cap P'$ be such a subinterval. By Lemma 1, we can pick a point, m , in $P'' \cap (c, d)$ so that $d(W, m) = d(B, m) = \frac{1}{2}$. We then pick $\eta > 0$ so that $[m - \eta, m + \eta] \subset [c, d]$ and, for any interval $K \subset [m - \eta, m + \eta]$ that contains m , $|\lambda(W \cap K) / \lambda(K) - \frac{1}{2}| < 0.00001$.

Let $P = [m - \eta, m + \eta] \cap P'$. For ease of calculation and notation, we will assume that $m = \frac{1}{2}$ and $\eta = \frac{3}{2}$. This gives:

- (1) $P \subset [-1, 2] = I$, $J = [0, 1]$ is the middle third of I ;
- (2) if $x \in P$ and $(x - \delta, x + \delta) \subset I$, then $\lambda(WB(x) \cap (0, \delta)) / \delta \leq 0.0001$;
- (3) for any interval $K \subset I$ that contains $\frac{1}{2}$,

$$|\lambda(W \cap K) / \lambda(K) - \frac{1}{2}| < 0.00001.$$

Let $W \cap (0, 1) = \bigcup G_n$ and $B \cap (0, 1) = \bigcup H_n$, the components of W and B in $(0, 1)$. If we order the intervals in $(\bigcup G_n) \cup (\bigcup H_n)$ by decreasing lengths, let ε be the first length so that the union of intervals of length greater than or equal to ε has measure greater than or equal to $\frac{1}{8}$. We can assume then, without loss of generality, that $\bigcup (G_n \mid \lambda(G_n) \geq \varepsilon)$ has measure at least $\frac{1}{16}$. Let this collection be $\{G_1, \dots, G_k\}$. Now, renumber these intervals in order from left to right. Let j be largest such that G_j is not a subset of the interval $(0.9999, 1)$. Then $\lambda(\bigcup_{n=1}^j G_n) \geq \frac{1}{16} - 0.0001$.

LEMMA 2. If a, b are in $J \cap P$, $a < b$, and $\lambda(W \cap (a, b)) \geq (b - a)\gamma$, then there is a $c > b$ in P so that $c - a \geq (b - a)(0.9999 + \gamma)$ and $\lambda(W \cap (a, c)) \geq (c - a)(\gamma - 0.00005)$.

PROOF. Let $c' = 2b - a$. Then $c' - a = 2(b - a) \geq (b - a)(0.9999 + \gamma)$ and $\lambda(W \cap (b, c')) \geq (c' - b)(\gamma - 0.0001)$ by (2). Thus $\lambda(W \cap (a, c')) \geq (c' - a)(\gamma - 0.00005)$. We distinguish three cases.

Case (i). If $c' \in P$, let $c = c'$.

Case (ii). If $c' \in W \cup C$, then c' is either contained in, or is a left endpoint of, a component of W . Let c be the right endpoint of this component, which is a point in P . This increases the relative measure of W so that $\lambda(W \cap (a, c)) \geq (c - a)(\gamma - 0.00005)$.

Case (iii). If $c' \in B$, let c be the left endpoint of the component of B containing c' , which is in P . Since $b \in P$, we have $(c' - c)/(c' - b) \leq 1 - (\gamma - 0.0001) = 1.0001 - \gamma$. So, $(c' - c) \leq (1.0001 - \gamma)(c' - b) = (1.0001 - \gamma)(b - a)$. Then $c - a = (c' - a) - (c' - c) \geq 2(b - a) - (1.0001 - \gamma)(b - a) = (b - a)(0.9999 + \gamma)$. Since $(c, c') \subset B$, $\lambda(W \cap (a, c)) = \lambda(W \cap (a, c')) \geq (c' - a)(\gamma - 0.00005) > (c - a)(\gamma - 0.00005)$.

This finishes the proof of the lemma.

Let (a, b) be one of the intervals $G_1, \dots, G_j$. We will show that seven successive applications of Lemma 2, with the left endpoint a fixed, produces an interval contained in J . For $a \leq \frac{1}{2}$, this is easy to see by the fact that W has relative measure roughly $\frac{1}{2}$ on any interval containing $\frac{1}{2}$. Consider $a > \frac{1}{2}$. Suppose n applications of Lemma 2 to (a, b) produces the interval (a, c) in J , but Lemma 2 applied to (a, c) extends beyond 1. Recall that $a < 0.9999$. Then $\lambda(W \cap (a, c)) \geq (c - a)(1 - 0.00005n)$, since $(a, b) \subset W$ and each application of the lemma can reduce the relative measure of W by at most 0.00005. We also have $\lambda(W \cap (\frac{1}{2}, a)) \geq 0.49999(a - \frac{1}{2})$ and $\lambda(W \cap (\frac{1}{2}, c)) \leq 0.50001(c - \frac{1}{2})$ by (3). This gives $\lambda(W \cap (a, c)) \leq 0.50001(c - \frac{1}{2}) - 0.49999(a - \frac{1}{2}) = 0.50001(c - a) + 0.00002(a - \frac{1}{2}) \leq 0.50001(c - a) + 0.00002$. Combining the two inequalities, we get $1 - 0.00005n \leq 0.50001 + 0.00002/(c - a)$. Since $2c - a > 1$ and $a < 0.9999$, $c - a \geq 0.0001/2 = 0.00005$. Thus $1 - 0.00005n \leq 0.50001 + 0.00002/0.00005 = 0.90001$, or $n > 1999$.

Now apply Lemma 2 seven times to G_1 to get the interval I_1 . Take the first G_i not in I_1 and apply Lemma 2 seven times to get the interval I_2 . Repeat this process through G_j to produce the intervals $I_1, \dots, I_m$. By straightforward calculation, each of these intervals is of length at least $(1.9999)^7 \varepsilon \geq 64\varepsilon$ and has relative measure of W at least $1 - (0.00005)(7) = 0.99965$. Since $\bigcup_{n=1}^j G_n \subset \bigcup_{n=1}^m I_n$, $\bigcup_{n=1}^m I_n$ has measure at least $\frac{1}{16} - 0.0001$.

Let $A = \bigcup_{n=1}^m I_n$, and $\beta = (B - A) \cap (0, 1)$. Observe that β is open since no component of B contains an endpoint of any of the finitely many components of A . We let T represent the triangle with vertices $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, \frac{1}{2})$, and $D \subset T$ is $\{(x, y) | y \in A\beta(x) \cup \beta A(x)\}$. Let H_i and I_k be components of β and A respectively. By intersecting the lines with slopes ± 1 through the endpoints of H_i and I_k , we obtain a rectangular region contained in D . Inside this region inscribe the parallelogram, D_{ik} , with two vertical sides that contain, as opposite vertices, the midpoints of the two shorter sides of the rectangle. Then D_{ik} projects onto the x -axis as an interval, V_{ik} , of length $\max(\lambda(H_i), \lambda(I_k))/2$, and the vertical section of D_{ik} is of length $\min(\lambda(H_i), \lambda(I_k))/2$. Thus, the area of D_{ik} is equal to $\lambda(H_i)\lambda(I_k)/4$ and $\lambda(V_{ik}) \geq \lambda(I_k)/2 \geq 32\varepsilon$. The area of $\bigcup D_{ik}$ is then greater than or equal to $\lambda(\beta)\lambda(A)/4 \geq (0.49999 - 0.00035)(\frac{1}{16} - 0.0001)/4 \geq 0.0076$. Let L_{ik} be the left half of D_{ik} . Then $\bigcup L_{ik}$ has area at least 0.0038.

Since $\bigcup L_{ik} \subset T$, some z in $(0, 1)$ must have a vertical section of $\bigcup L_{ik}$ of linear measure at least 0.0038. Then every x in $(z, z + 16\epsilon)$ has a vertical section of $\bigcup D_{ik}$ of measure at least 0.0038 also. Let $E = W \cap A$ and $F = A - W$. Then, for each x in $(z, z + 16\epsilon)$, we have

$$\begin{aligned} \lambda(WB(x) \cup BW(x)) &\geq \lambda(EB(x)) + \lambda(BE(x)) \\ &\geq \lambda(\{h | (x, h) \in D\}) - \lambda(FB(x)) - \lambda(BF(x)) \\ &\geq 0.0038 - 2\lambda(F) \\ &\geq 0.0038 - 0.00035 - 0.00035 = 0.0031. \end{aligned}$$

LEMMA 3. For x in $(0, 1)$ and $z = \max(x, 1 - x)$, $\lambda(WB(x) \cap (0, z)) \geq \lambda(BW(x) \cap (0, z)) - 0.00004z$.

PROOF. We prove the case $z = x \geq \frac{1}{2}$, the other case being symmetric. Let μ_1, μ_2 , and μ_3 be the relative measures of B in the intervals $(x, 2x)$, $(0, x)$, and $(0, 2x)$ respectively. Then $\mu_3 = (\mu_2 + \mu_1)/2$ gives $\mu_1 = 2\mu_3 - \mu_2 \geq 2(0.49999) - 0.50001 = 0.49997$ by (3). Then $\lambda((WB(x) \cup BB(x)) \cap (0, x)) = \lambda(B \cap (x, 2x)) \geq 0.49997x$. Also, (3) gives $\lambda((WB(x) \cup WW(x)) \cap (0, x)) = \lambda(W \cap (0, x)) \geq 0.49999x$. Adding these two inequalities and using the fact that $WB(x) \cup WW(x) \cup BB(x) \cup BW(x)$ has full measure in $(0, x)$, we get $\lambda(WB(x) \cap (0, x)) + x - \lambda(BW(x) \cap (0, x)) \geq 0.99996x$, which is the desired inequality. This finishes the proof of the lemma.

We now establish an estimate for the longest possible component of W or B in $(0, 1)$.

LEMMA 4. If (a, b) is a component of $W \cup B$ in $(0, 1)$, then $b - a \leq 0.000025$.

PROOF. Since no component can contain $\frac{1}{2}$, it suffices to prove the case (a, b) a component of W contained in $(\frac{1}{2}, 1)$. Let $\mu_1 = 1$, μ_2 , and μ_3 be the relative measures of W in the intervals (a, b) , $(\frac{1}{2}, a)$, and $(\frac{1}{2}, b)$ respectively. Then $(b - \frac{1}{2})\mu_3 = (a - \frac{1}{2})\mu_2 + (b - a)$. Rearrangement gives $(b - a) = (a - \frac{1}{2})(\mu_3 - \mu_2)/(1 - \mu_3) \leq 0.5(0.00002)/(1 - 0.50001) \leq 0.000025$. This finishes the proof of the lemma.

Let c be the midpoint of $V = (z, z + 16\epsilon)$, and U the union of all G_n and H_n of length greater than ϵ . Then $\lambda(U) < \frac{1}{8}$. Without loss of generality, we may assume $0 < c \leq \frac{1}{2}$. Define $U' = \{2x - c | x \in U\}$. Since $\lambda(U') < \frac{1}{4}$, $\lambda(U \cup U') < \frac{3}{8}$. We can then pick $y \in [\frac{9}{16}, \frac{15}{16}] - (U \cup U')$. Let $x = (y + c)/2$. Since $2x - c = y \notin U'$, $x \notin U$. Thus, there are points s and t in P within ϵ of x and y respectively. To see this, consider y first. If $y \in P$, let $t = y$. If $y \notin P$, then y is in the closure of some G_n or H_n . By Lemma 4 and the fact that $y \notin U$, this interval can have length no greater than $\epsilon \leq 0.000025$. Therefore, both endpoints of this interval are in $(0, 1)$ and at least one of them is in P and is designated t . The points x and s are handled the same way. Then, for $r = 2s - t$, we have $|r - c| = |(2s - t) - c| = |2x + 2s - 2x - y + y - t - c| = |(2x - y - c) + 2(s - x) + (y - t)| \leq 0 + 2|s - x| + |y - t| \leq 3\epsilon$. This gives $r \in V$ and $\lambda((WB(r) \cup BW(r)) \cap (0, r')) \geq 0.0031$, where $r' = \min(r, 1 - r)$.

Our last step is to show that $WB(s)$ or $WB(t)$ violates (2). We define four maps as follows. Let $\rho_1(h) = h + s - r$ on $(BW(r) \cup WB(r)) \cap (0, r')$, $\rho_2(h) = h$ on $(BW(r) \cup WB(r)) \cap (0, r')$, $\rho_3(h) = h - s + r$ on $(BW(r) \cup WB(r)) \cap [s - r, r')$, and $\rho_4(h) = -(h - s + r)$ on $(BW(r) \cup WB(r)) \cap (0, s - r)$. Clearly, each ρ_i is one-to-one and measure preserving on its domain. If $h - s + r \geq 0$ then

$(r, h), (s, \rho_1(h)), (t, \rho_2(h))$, and $(s, \rho_3(h))$ are the vertices of a square. If $h - s + r < 0$, the fourth vertex $(s, h - s + r)$ is reflected above the x -axis to $(s, \rho_4(h))$.

Consider $h \in BW(r)$. Then $\rho_1(h) \in BW(s)$, $\rho_2(h) \in WB(t)$, $\rho_3(h) \in BW(s)$, $\rho_4(h) \in WB(s)$, $\rho_1(h) \in BQ(s)$, or $\rho_2(h) \in QB(t)$. A similar statement corresponds to $h \in WB(r)$. For $r' = \min(r, 1 - r)$, $s'' = \max(s, 1 - s)$, and $t'' = \max(t, 1 - t)$, define the following subsets of $(BW(r) \cup WB(r)) \cap (0, r')$. Let

$$\begin{aligned} S_1 &= \{h | \rho_1(h) \in (BW(s) \cup WB(s)) \cap (0, s'')\}, \\ S_2 &= \{h | \rho_2(h) \in (BW(t) \cup WB(t)) \cap (0, t'')\}, \\ S_3 &= \{h | \rho_3(h) \in (BW(s) \cup WB(s)) \cap (0, s'')\}, \\ S_4 &= \{h | \rho_4(h) \in (BW(s) \cup WB(s)) \cap (0, s'')\}, \\ S_5 &= \{h | \rho_1(h) \in BQ(s) \cup WQ(s) \text{ or } \rho_2(h) \in QB(t) \cup QW(t)\}. \end{aligned}$$

Since $\lambda(Q) = 0$, we have $\lambda(S_5) = 0$. Since $(BW(r) \cup WB(r)) \cap (0, r') = \bigcup_{i=1}^5 S_i$, one of the sets S_1, S_2, S_3 , or S_4 has measure at least $0.0031/4 > 0.00075$. Therefore, one of $(BW(s) \cup WB(s)) \cap (0, s'')$ or $(BW(t) \cup WB(t)) \cap (0, t'')$ has measure at least 0.00075 . We assume the former set has measure at least $0.00075 \geq 0.00075s''$.

By (2), $\lambda(WB(s) \cap (0, s'')) \leq 0.0001s''$, so $\lambda(BW(s) \cap (0, s'')) \geq 0.00065s''$. But then Lemma 3 gives $\lambda(WB(s) \cap (0, s'')) \geq 0.00065s'' - 0.00004s'' = 0.00061s''$, contradicting (2). This finishes the proof of the theorem.

An application to the approximate symmetric derivative.

THEOREM 2. *If f is continuous on $[a, b]$ and $\underline{f}_{ap}^{(1)} \geq 0$ on (a, b) , then f is nondecreasing on $[a, b]$.*

PROOF. For $\varepsilon > 0$, let $g(x) = f(x) + \varepsilon x$. Then $\underline{g}_{ap}^{(1)} > 0$ on (a, b) . Let y be any number in the interior of the range of g . It is easy to see that $W = g^{-1}(y, \infty)$ and $B = g^{-1}(-\infty, y)$ satisfy the hypotheses of Theorem 1. Then g must be nondecreasing, since if $c < d$ and $g(c) > g(d)$, we can pick any y between $g(c)$ and $g(d)$ and get $WB((c+d)/2) \neq \emptyset$, contradicting Theorem 1. Since ε is arbitrary, f is nondecreasing. This finishes the proof of the theorem.

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SAN BERNARDINO, CALIFORNIA 92407