RADII PROBLEMS FOR SECTIONS OF CONVEX FUNCTIONS

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ABSTRACT. A classical theorem of Szegő states that the sections $f_n(z) = z + \sum_{k=2}^n a_k z^k$ of a convex function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ must be convex for $|z| < \frac{1}{4}$. We determine disks $|z| < r_n$ in which f_n is starlike and starlike of a positive order. Our proofs rely on some properties of convolutions.

1. Introduction. The *convolution* or *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Denote by S^* the family consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and starlike in $\Delta = \{|z| < 1\}$ and by K the subfamily of functions convex in Δ . In 1958 Pólya and Schoenberg conjectured in [3] that the convolution of two functions convex in Δ is also convex in Δ . Since $f * g = h$ is equivalent to $f * zg' = zh'$, the well-known relationship $f \in K \Leftrightarrow zf' \in S^*$ furnishes us with an alternate statement of the Pólya-Schoenberg conjecture. This conjecture was proved in 1973 by Ruscheweyh and Sheil-Small.

THEOREM A [5]. *If $f \in K$ and $g \in K$ ($g \in S^*$), then $f * g \in K$ ($f * g \in S^*$).*

Goodman and Schoenberg in [1] applied Theorem A to give a much simpler proof of the 1928 classical theorem of Szegő on the sections of convex functions.

THEOREM B [7]. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$, then $f_n(z) = z + \sum_{k=2}^n a_k z^k$ is convex in $|z| < \frac{1}{4}$ for all n and $\frac{1}{4}$ is the largest radius with this property.*

Goodman and Schoenberg first observed that it was sufficient to look at sections of the convex function $g(z) = \sum_{k=1}^{\infty} z^k = z/(1-z)$. For if $g_n(z) = z + \sum_{k=2}^n z^k$ is convex in $|z| < t < 1$, then $g_n(tz)/t \in K$ and hence $h_n(z) = f(z) * g_n(tz)/t = z + \sum_{k=2}^n a_k t^{k-1} z^k \in K$. Consequently, $th_n(z/t) = f_n(z)$ is convex in $|z| < t$. Their proof was concluded when $t = \frac{1}{4}$ was shown to be admissible.

An identical argument shows that any section of a convex function is starlike in $|z| < t$ whenever $g_n(z)$ is. In this note, we find disks $|z| < r_n$ in which the sections f_n of a convex function f are starlike. Analogous results are obtained for functions of positive order. Finally, we examine what can happen when the sections are replaced

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by more general functions containing infinitely many nonvanishing coefficients of f . A conjecture that would then generalize Szegő's Theorem B is given.

In the sequel we shall assume, unless otherwise stated, that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$, that $g(z) = z + \sum_{k=2}^{\infty} z^k$, and that f_n and g_n are the sections of f and g respectively.

2. Radius of starlikeness for sections of K .

THEOREM 1. *If $f \in K$, then f_n is starlike in $|z| < (1/2n)^{1/n}$ for all n .*

PROOF. It suffices to prove the theorem for

$$g_n(z) = z + \sum_{k=2}^n z^k = (z - z^{n+1})/(1 - z).$$

We have

$$(1) \quad \frac{zg'_n(z)}{g_n(z)} = \frac{1}{1-z} - \frac{nz^n}{1-z^n}$$

so that for $|z| = r$,

$$(2) \quad \operatorname{Re} \left\{ \frac{zg'_n(z)}{g_n(z)} \right\} \geq \frac{1}{1+r} - \frac{nr^n}{1-r^n}.$$

We need only show that the right-hand side of (2) is nonnegative or, equivalently, that

$$(3) \quad 1 - (n+1)r^n - nr^{n+1} \geq 0$$

for $r \leq (1/2n)^{1/n}$. But when $r = (1/2n)^{1/n}$, (3) is equivalent to $r = (1/2n)^{1/n} \leq 1 - 1/n$, which is true for all $n \geq 2$.

Since the sequence $(1/2n)^{1/n}$ increases with n , a consequence of Theorem 1 is

COROLLARY 1. *If $f \in K$, then f_n is starlike in $|z| < \frac{1}{2}$ and $\frac{1}{2}$ is the largest radius with this property.*

Note for $g_2(z) = z + z^2$ that $g'_2(z) = 1 + 2z = 0$ at $z = -\frac{1}{2}$, which means that the radius of univalence of sections of a convex function is also $\frac{1}{2}$.

COROLLARY 2. *If $f \in K$, then f_n is starlike in $|z| < \sqrt{23/71} \approx 0.569$ for $n \geq 3$ and $\sqrt{23/71}$ is the largest radius with this property.*

PROOF. From (1) we have for $z = re^{i\theta}$,

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zg'_3(z)}{g_3(z)} \right\} &= \operatorname{Re} \left\{ \frac{1}{1-z} - \frac{3z^3}{1-z^3} \right\} = \operatorname{Re} \left\{ \frac{1+2z+3z^2}{1+z+z^2} \right\} \\ &= \frac{(1-2r^2+3r^4) + (3r+5r^3)\cos\theta + 8r^2\cos^2\theta}{|1+z+z^2|^2}. \end{aligned}$$

For each r the numerator of this last expression, which we denote by $h(r, \theta)$, has a minimum at $\theta = \pi$ or at

$$\theta = \theta_0 = \cos^{-1} \left(-\frac{3+5r^2}{16r} \right).$$

Since $h(r, \pi) \geq 0$ for $0 \leq r \leq 1$ and $h(r, \theta_0) = (23 - 94r^2 + 71r^4)/32 = 0$ for $r = \sqrt{23/71}$, the bound cannot be increased for $n = 3$. The result follows from Theorem 1 upon observing for $n \geq 4$ that $(1/2n)^{1/n} \geq (1/8)^{1/4} > (23/71)^{1/2}$.

REMARKS. For n even, we may set $z = -r$ in (1) to conclude that the radius of starlikeness of g_n is the unique positive r for which equality holds in (3). There does not appear to be such a "nice" polynomial equation from which to determine the radius of starlikeness of sections of a convex function when n is odd.

In 1941, Robertson proved [4] that sections of a function in S^* must be starlike in $|z| < 1 - (4 \log n)/n$ for $n > N$. He further showed that sections of the Koebe function $z/(1-z)^2$ are starlike in $|z| < 1 - (3 \log n)/n$ and that the constant 3 cannot be lowered. But f is convex whenever zf' is starlike, so that we may translate Robertson's result to its convex analog. In view of the more recent developments (Theorem A), Robertson essentially proved

THEOREM 2. *If $f \in K$, then f_n is convex in $|z| < 1 - (3 \log n)/n$ for n sufficiently large. The constant 3 cannot be lowered.*

REMARK. Since

$$\left(1 - \frac{a \log n}{n}\right)^n \sim \frac{1}{n^a} \quad (n \rightarrow \infty),$$

we see for any $a > 1$ and n sufficiently large that $(1/an)^{1/n} > 1 - (a \log n)/n$. To put Theorem 1 in a form analogous to Theorem 2, we have for any $\varepsilon > 0$ and $n > N(\varepsilon)$ that f_n is starlike in $|z| < 1 - ((1 + \varepsilon) \log n)/n$.

3. Positive order. Denote by $S^*(\alpha)$, $0 \leq \alpha < 1$, the family consisting of analytic functions $f(z) = z + \dots$ that satisfy $\operatorname{Re}\{zf'/f\} > \alpha$, $z \in \Delta$, and by $K(\alpha)$ the subfamily for which $\operatorname{Re}\{1 + zf''/f'\} > \alpha$, $z \in \Delta$. Such functions are said to be, respectively, *starlike of order α* and *convex of order α* . The work of Ruscheweyh and Sheil-Small [5] actually shows that if g in Theorem A is in $K(\alpha)$ or $S^*(\alpha)$, then so is $f * g$.

The first important result about functions of positive order is due to Marx [2] and Strohcker [6] who independently proved that $K \subset S^*(\frac{1}{2})$. In fact, they showed that $\operatorname{Re}\{zf'/f\} \geq 1/(1+r)$, $f \in K$, with equality for $f(z) = z/(1-z)$ at $z = -r$. We may thus conclude that f in K is starlike of order α for $|z| < (1-\alpha)/\alpha$ when $\frac{1}{2} \leq \alpha < 1$. The more general form of the theorem of Ruscheweyh and Sheil-Small will now be applied to find disks in which sections of f are starlike of order α .

THEOREM 3. *If $f \in K$, then f_n is starlike of order α in $|z| < (1-\alpha)/(2-\alpha)$ and $(1-\alpha)/(2-\alpha)$ is the largest radius with this property.*

PROOF. In view of (2), it suffices to show that

$$(5) \quad \frac{1}{1+r} - \frac{nr^n}{1-r^n} \geq \alpha$$

for $r \leq (1-\alpha)/(2-\alpha)$. Since equality in (5) holds for $r = (1-\alpha)/(2-\alpha)$ and $n = 2$, we need only show that $nr^n/(1-r^n)$ is a decreasing function of n for $r = (1-\alpha)/(2-\alpha)$. But $(n+1)r^{n+1}/(1-r^{n+1}) \leq nr^n/(1-r^n)$ is equivalent to $n - (n+1)r + r^{n+1} \geq 0$, which follows upon noting that the left-hand side is a decreasing function of r (≤ 1). The function $g_2(z) = z + z^2$ is extremal.

In light of the Marx and Strohcker result, it is no surprise that the value $\frac{1}{2}$ plays a distinguishing role when finding disks for which sections of convex functions are starlike of order α . We show that the f_n must be starlike of order α in disks $|z| < r_n \rightarrow 1$ if and only if $\alpha \leq \frac{1}{2}$.

THEOREM 4. *Suppose $f \in K$.*

- (i) *If $0 \leq \alpha < \frac{1}{2}$, then f_n is starlike of order α in $|z| < ((1 - 2\alpha)/2n)^{1/n}$ for all n .*
- (ii) *The function f_n is starlike of order $\frac{1}{2}$ in $|z| < (4/9n^2)^{1/n}$ for all n .*
- (iii) *If $\frac{1}{2} < \alpha < 1$, then g_{2n} is not starlike in $|z| < (1 - \alpha)/\alpha < 1$ for any n .*

PROOF. We rewrite (5) as

$$(6) \quad (1 - \alpha) - \alpha r - (n + 1 - \alpha)r^n - (n - \alpha)r^{n+1} \geq 0.$$

To prove (i), set $r = ((1 - 2\alpha)/2n)^{1/n}$ in (6) to obtain

$$r = \left(\frac{1 - 2\alpha}{2n} \right)^{1/n} \leq \frac{n - (1 - \alpha)(1 - 2\alpha)}{n - \alpha(1 - 2\alpha)},$$

which is equivalent to

$$h(\alpha, n) = \left(\frac{n - (1 - \alpha)(1 - 2\alpha)}{n - \alpha(1 - 2\alpha)} \right) \left(\frac{2n}{1 - 2\alpha} \right)^{1/n} \geq 1.$$

Since $h(\alpha, n) \geq h(0, n) = (1 - 1/n)(2n)^{1/n} \geq h(0, 2) = 1$, (i) follows. Setting $\alpha = \frac{1}{2}$ and $r = (4/9n^2)^{1/n}$ in (6) leads to $4/9n^2 \leq (1 - 16n/(9n^2 + 8n - 4))^n$, which is true for all n with equality when $n = 2$. This proves (ii). Finally, if $\frac{1}{2} < \alpha < 1$ then the left-hand side of (6) is less than $1 - \alpha - \alpha r$, which vanishes for $r = (1 - \alpha)/\alpha$.

REMARKS. While not sharp for every n , Theorem 4 is at least of the right order of magnitude. For $r = (A/n)^{1/n}$, $A > (1 - 2\alpha)/2$, we may let $n \rightarrow \infty$ in (6) to get $1 - 2\alpha - 2A < 0$. Hence g_{2n} is not starlike of order α ($< \frac{1}{2}$) for n sufficiently large. Similarly, we can show that g_{2n} is not starlike of order $\frac{1}{2}$ for n large when $|z| < (\varepsilon/n^\beta)^{1/n}$, $\varepsilon > 0$ and $\beta < 2$.

Turning now to disks in which sections of convex functions are convex of order α , we note first that $\operatorname{Re}\{1 + zf''/f'\} \geq (1 - r)/(1 + r)$, $f \in K$, so that f is convex of order α when $|z| < (1 - \alpha)/(1 + \alpha)$. The next theorem reduces to Szeg's Theorem when $\alpha = 0$. The proof uses the methods of Goodman and Schoenberg [1].

THEOREM 5. *If $f \in K$, then f_n is convex of order α in $|z| < (1 - \alpha)/2(2 - \alpha)$ and $(1 - \alpha)/2(2 - \alpha)$ is the largest radius with this property.*

PROOF. For $g_n(z) = (z - z^{n+1})/(1 - z)$, we have

$$1 + \frac{zg_n''(z)}{g_n'(z)} = \frac{n(n+1)z^n(z-1)}{1 - (n+1)z^n + nz^{n+1}} + \frac{1+z}{1-z}.$$

It suffices to show, for $|z| = (1 - \alpha)/2(2 - \alpha)$, that

$$(7) \quad \left| \frac{n(n+1)z^n(z-1)}{1 - (n+1)z^n + nz^{n+1}} \right| \leq \operatorname{Re} \left\{ \frac{1+z}{1-z} \right\} - \alpha.$$

Since the right-hand side of (7) is minimized when z is real and negative, we need only show that

$$(8) \quad \frac{n(n+1)|z|^n(|z|+1)}{1-(n+1)|z|^n-n|z|^{n+1}} \leq \frac{3-\alpha}{5-3\alpha} - \alpha = \frac{3(1-\alpha)^2}{5-3\alpha}.$$

Substituting in for $|z|$, a computation shows that (8) is equivalent to

$$\frac{(5-3\alpha)^2}{6(1-\alpha)^2(2-\alpha)} \leq \frac{2^n(2-\alpha)^n}{n(n+1)(1-\alpha)^n} - \frac{1}{n} - \frac{1-\alpha}{2(n+1)(2-\alpha)} = j(\alpha, n).$$

The result follows upon noting that $j(\alpha, n)$ is an increasing function of n , α fixed, with $j(\alpha, 2) = (5-3\alpha)^2/6(1-\alpha)^2(2-\alpha)$. The function $g_2(z)$ is extremal.

4. More general sequences. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$ and $\{a_{k_j}\}$ is any subsequence of $\{a_k\}$, finite or infinite, Theorem A may again be applied to show that $F_k(z) = z + \sum_{j=1}^{\infty} a_{k_j} z^{k_j}$ is starlike or convex whenever $G_k(z) = z + \sum_{j=1}^{\infty} z^{k_j}$ is. In the special case that $k_j = k$ for $1 \leq j \leq n$ ($n = 2, 3, \dots$), our functions reduce to the sections previously investigated. To solve extremal problems, it behooves us to find as much information as possible about G_k . To illustrate with an example, if we set $k_j = jk + 1$ then $G_k(z) = z + \sum_{j=1}^{\infty} z^{jk+1} = z/(1-z^k)$. This leads to

THEOREM 6. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$, then*

$$F_k(z) = z + \sum_{j=1}^{\infty} a_{jk+1} z^{jk+1} \quad (k = 2, 3, \dots)$$

is starlike in $|z| < (1/(k-1))^{1/k}$. The bound is sharp for every k .

PROOF. For $G_k(z) = z/(1-z^k)$, we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z G'_k(z)}{G_k(z)} \right\} &= \operatorname{Re} \left\{ \frac{1 + (k-1)z^k}{1-z^k} \right\} = \frac{1 + (k-2)r^k \cos k\theta - (k-1)r^{2k}}{|1-z^k|^2} \\ &\geq \frac{1 - (k-2)r^k - (k-1)r^{2k}}{|1-z^k|^2}. \end{aligned}$$

Since the numerator of this last expression vanishes when $r = (1/(k-1))^{1/k}$, the proof is complete.

Now $(1/(k-1))^{1/k}$ attains its minimum when $k = 5$. So with the notation of the theorem, we obtain

COROLLARY 1. *If $f \in K$, then F_k is starlike in $|z| < (1/4)^{1/5} \approx 0.758$ for all $k = 2, 3, \dots$.*

One can obtain similar results about convexity. For instance, a simple computation shows the radius of convexity of $G_2(z) = z/(1-z^2)$ to be $\sqrt{2} - 1$, which yields

COROLLARY 2. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$, then $F_2(z) = z + \sum_{j=1}^{\infty} a_{2j+1} z^{2j+1}$ is convex in $|z| < \sqrt{2} - 1 \approx 0.414$.*

The function $g_2(z) = z + z^2$ has been extremal for most of our results on sections of a convex function. We close with a conjecture that would establish g_2 as extremal in a more general setting, which incorporates Szegő's Theorem B and Corollary 1 of Theorem 1 as special cases.

CONJECTURE. If $z + \sum_{k=2}^{\infty} a_k z^k \in K$ and $\{a_{k_j}\}$ is any (finite or infinite) subsequence of $\{a_k\}$, then $z + \sum_{j=1}^{\infty} a_{k_j} z^{k_j}$ is convex in $|z| < \frac{1}{4}$ and starlike in $|z| < \frac{1}{2}$.

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