COMPARISON THEOREMS FOR EIGENVALUE PROBLEMS FOR n TH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. We give a comparison theorem for eigenvalues for a $(k, n - k)$ -conjugate boundary value problem for the systems $(-1)^{n-k}Ly = \lambda P(t)y$ and $(-1)^{n-k}Lz = \Lambda Q(t)z$, where $P(t)$ and $Q(t)$ are continuous $m \times m$ matrix functions. We assume that the corresponding scalar equation $Lx = 0$ is $(j, n - j)$ -disconjugate for $k - 1 \leq j \leq n - 1$. A special case of this is when $Lx = 0$ is disconjugate; our results are new even in this case.

1. Introduction. We are concerned with proving comparison theorems for the eigenvalue problems

$$(1) \quad (-1)^{n-k}Ly = \lambda P(t)y$$

and

$$(2) \quad (-1)^{n-k}Lz = \Lambda Q(t)z$$

subject to the boundary conditions

$$(3) \quad \begin{aligned} y^{(i)}(a) &= 0, & 0 \leq i \leq k - 1, \\ y^{(i)}(b) &= 0, & 0 \leq i \leq n - k - 1. \end{aligned}$$

Here, $1 \leq k \leq n - 1$, P and Q are continuous $m \times m$ matrix functions on $[a, b]$, and

$$Ly(t) = \sum_{i=0}^n a_i(t)y^{(i)}(t)$$

where $a_i(t)$, $0 \leq i \leq n$, are continuous scalar functions on $[a, b]$ with $a_n(t) > 0$ on $[a, b]$.

In this paper, we prove existence and comparison theorems for the least positive eigenvalues of (1), (3) and (2), (3), respectively. The methods used depend on the theory of operators on a cone; the books of Krasnosel'skiĭ [6] and Deimling [1] are good references. Related papers include those of Gentry and Travis [2], Keener and Travis [4, 5], Kreith [7], Schmitt and Smith [10], Smith [11], Tomastik [13], and the authors [3].

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Typically, “positivity” conditions are placed on $P(t)$ and $Q(t)$ in order that related operators be u_0 -positive. This leads to existence and comparison theorems for the least positive eigenvalues. In addition, we show how these requirements can be relaxed and still obtain existence and comparison results.

2. Auxiliary results. We say that the scalar equation

$$Lx \equiv \sum_{i=0}^n a_i(t)x^{(i)} = 0$$

is $(j, n-j)$ -disconjugate on $[a, b]$ provided there is no nontrivial solution $x(t)$ and numbers $a \leq \alpha < \beta \leq b$ such that

$$\begin{aligned} x^{(i)}(\alpha) &= 0, & 0 \leq i \leq j-1, \\ x^{(i)}(\beta) &= 0, & 0 \leq i \leq n-j-1. \end{aligned}$$

We say that $Lx = 0$ is *disconjugate* on an interval I if no nontrivial solution has n or more zeros (counting multiplicities) on I .

It is well known [8] that $Lx = 0$ is $(j, n-j)$ -disconjugate on an interval I if L is of the form

$$Lx = Mx + r(t)x, \quad (-1)^{n-j}r(t) \geq 0 \quad \text{on } I$$

where $Mx = 0$ is any n th order linear disconjugate equation on I (with leading coefficient positive).

We will assume that the scalar equation $Lx = 0$ is $(j, n-j)$ -disconjugate on $[a, b]$ for $j = k-1, \dots, n-1$ (if $k = 1$, we assume $Lx = 0$ is disconjugate on $[a, b]$). It is well known that if $Lx = 0$ is disconjugate on $[a, b]$ then this condition is satisfied. If $k > 2$, then the converse need not hold. Our results appear to be new even in the disconjugate case. By Theorem 11 of [9], it follows that the Green's function $G(t, s)$ for the scalar problem

$$\begin{aligned} (-1)^{n-k}Lx &= h(t) \\ x^{(i)}(a) &= 0, & 0 \leq i \leq k-1, \\ x^{(i)}(b) &= 0, & 0 \leq i \leq n-k-1, \end{aligned}$$

exists, and satisfies the positivity condition

$$(4) \quad G(t, s) > 0, \quad t, s \in (a, b).$$

We now give several definitions as given in [6] which we will use in this paper. Let $\mathcal{B}$ be a Banach space. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a *cone* provided

- (i) if $u, v \in \mathcal{P}$ then $\alpha u + \beta v \in \mathcal{P}$ for all $\alpha, \beta \geq 0$,
- (ii) if $u, -u \in \mathcal{P}$ then $u = 0$.

A cone $\mathcal{P}$ is said to be *reproducing* provided every $w \in \mathcal{B}$ can be written $w = u - v$ for some $u, v \in \mathcal{P}$. A cone $\mathcal{P}$ is *solid* provided its interior $\mathcal{P}^\circ$ is nonempty. Every solid cone is reproducing (see [6]). If $\mathcal{P}$ is a cone and $u, v \in \mathcal{B}$ then we write $u \leq v$ with respect to $\mathcal{P}$ provided $v - u \in \mathcal{P}$. If M and N are operators on $\mathcal{B}$, then we write $M \leq N$ with respect to $\mathcal{P}$ provided $Mu \leq Nu$ for all $u \in \mathcal{P}$. Finally we say that a bounded linear operator M on $\mathcal{B}$ is u_0 -positive with respect to the cone $\mathcal{P}$

provided $u_0 \in \mathcal{P}$ and for every nonzero $u \in \mathcal{P}$ there exist positive numbers k_1 and k_2 (which in general depend on u) such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

We will make use of the following results which we state here for easy reference. The first is Theorem 2.5 of Krasnosel'skiĭ [6].

THEOREM 1. *Let L be a linear compact operator which leaves the cone $\mathcal{P}$ invariant. Assume there is a $u \in \mathcal{B}$ and an $\varepsilon > 0$ with $-u \notin \mathcal{P}$ and $u = v - w$ for some $v, w \in \mathcal{P}$ such that $L^p u \geq \varepsilon u$ for some natural number p . Then L has at least one eigenvector x_0 in $\mathcal{P}$: $Lx_0 = \lambda_0 x_0$, where the eigenvalue λ_0 satisfies $\lambda_0 \geq \sqrt[p]{\varepsilon}$.*

COROLLARY 1. *Assume in addition to the hypotheses of Theorem 1 that $\mathcal{P}$ is reproducing. Then one of the eigenvalues of L corresponding to an eigenvector in $\mathcal{P}$ is an upper bound for the modulus of the eigenvalues of L .*

Theorems 2.5, 2.10, 2.11, and 2.13 of Krasnosel'skiĭ [6] give the following theorem.

THEOREM 2. *Let $\mathcal{P}$ be a reproducing cone. If L is a compact u_0 -positive linear operator then L has an essentially unique eigenvector in $\mathcal{P}$ and the corresponding eigenvalue is simple, positive, and larger than the modulus of any other eigenvalue.*

The following theorem appears in Keener and Travis [4, Theorem 2.3] and is a generalization of Theorem 2.3 of Travis [14].

THEOREM 3. *Let L and M be bounded linear operators and assume that at least one of the operators is u_0 -positive. If $L \leq M$ and*

$$\begin{aligned} Lu_1 &\geq \lambda_1 u_1 & (0 \neq u_1 \in \mathcal{P}, \lambda_1 > 0), \\ Mu_2 &\leq \lambda_2 u_2 & (0 \neq u_2 \in \mathcal{P}, \lambda_2 > 0), \end{aligned}$$

then $\lambda_1 \leq \lambda_2$, and if $\lambda_1 = \lambda_2$ then u_1 is a scalar multiple of u_2 .

For related statements of these results of Krasnosel'skiĭ, see Theorems 2.1–2.4 [10].

3. Main results. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{y \in C^n([a, b], R^m) : y^{(i)}(a) = 0, 0 \leq i \leq k-1, \\ y^{(i)}(b) = 0, 0 \leq i \leq n-k-1\}$$

where

$$\|y\| = \max\{|y^{(i)}(t)| : t \in [a, b], 0 \leq i \leq n\}$$

and $|\cdot|$ is the Euclidean norm. Let $\mathcal{K}$ be a solid cone in R^m , and define the cone $\mathcal{P}$ in $\mathcal{B}$ by

$$\mathcal{P} = \{y \in \mathcal{B} : y(t) \in \mathcal{K}, a \leq t \leq b\}.$$

Except for a few of our general results, we will be interested in the special type of solid cone $\mathcal{K}_1$ defined below. As in Tomastik's paper [13], let $I, J \subseteq \{1, \dots, m\}$ be such that $I \cup J = \{1, \dots, m\}$ and $I \cap J = \emptyset$ ($I = \emptyset$ or $J = \emptyset$ is permissible). Define the cone $\mathcal{K}_1 \subseteq R^m$ by

$$\mathcal{K}_1 = \{y = (y_1, \dots, y_m) : y_i \geq 0 \text{ if } i \in I, y_i \leq 0 \text{ if } i \in J\}.$$

Throughout this paper assume $\delta_i = 1$ if $i \in I$, and $\delta_i = -1$ if $i \in J$. Note that $y \in K_1$ iff $\delta_i y_i \geq 0$, $0 \leq i \leq m$. Then define the cone $\mathcal{P}_1$ in $\mathcal{B}$ by

$$\mathcal{P}_1 = \{y \in \mathcal{B} : y(t) \in K_1, t \in [a, b]\}.$$

We remark that it is not true that the interior of the cone $\mathcal{P}$ is the set $\{y \in \mathcal{B} : y(t) \in K^\circ \text{ for } t \in (a, b)\}$, although this type of oversight appears in some of the literature. Keep this in mind when reading the proof of Theorem 4 below.

Define operators M and N on $\mathcal{B}$ by

$$\begin{aligned} Mu(t) &= \int_a^b G(t, s)P(s)u(s) ds, \quad t \in [a, b], \\ Nu(t) &= \int_a^b G(t, s)Q(s)u(s) ds, \quad t \in [a, b]. \end{aligned}$$

It is clear that M and N are linear, and a standard argument using Ascoli's Theorem shows that they are compact operators.

REMARK 1. If $\lambda \neq 0$ is an eigenvalue of M with corresponding eigenvector y , then y is an eigenfunction of (1), (3) with corresponding eigenvalue $1/\lambda$. Note that $\lambda = 0$ is not an eigenvalue for (1), (3) since $Lx = 0$ is $(k, n - k)$ -disconjugate on $[a, b]$.

THEOREM 4. Assume $Q(t)K \subseteq K$ for $t \in [a, b]$, and for each nontrivial u in $\mathcal{P}$, there exists $t_u \in [a, b]$ such that $Q(t_u)u(t_u) \in K^\circ$. Then the boundary value problem (2), (3) has a smallest positive eigenvalue Λ_0 , and Λ_0 is smaller than the modulus of any other eigenvalue of (2), (3). Further, there is an essentially unique eigenfunction $z_0(t)$ corresponding to Λ_0 and either $z_0 \in \mathcal{P}^\circ$ or $-z_0 \in \mathcal{P}^\circ$.

PROOF. We will show that $N : \mathcal{P} \setminus \{0\} \rightarrow \mathcal{P}^\circ$. Let $0 \neq u \in \mathcal{P}$ and define h by

$$h(t) = Nu(t) = \int_a^b G(t, s)Q(s)u(s) ds.$$

It follows (see Lemma 2.3 of [5]) from (4) and the assumptions on $Q(t)$ that

$$h(t) \in K^\circ, \quad t \in (a, b).$$

As we pointed out earlier, this does not imply that $h \in \mathcal{P}^\circ$. To see that $h \in \mathcal{P}^\circ$, we will assume that $h \notin \mathcal{P}^\circ$ and show that this leads to a contradiction.

Since $h \notin \mathcal{P}^\circ$, there is a sequence $\{y^j\} \subseteq \mathcal{B}$ such that $y^j \rightarrow h$ in $\mathcal{B}$ and $y^j \notin \mathcal{P}$, $j \geq 1$. Then there is a $t_j \in [a, b]$ such that $y^j(t_j) \notin K$ for each $j \geq 1$. Without loss of generality, we can assume $t_j \rightarrow t_0$ where $t_0 \in [a, b]$. If $t_0 \in (a, b)$, then $y^j(t_j) \rightarrow h(t_0) \in K^\circ$ implies that for sufficiently large j , $y^j(t_j) \in K^\circ$ which is a contradiction.

The only possibilities that remain are that $t_0 = a$ or $t_0 = b$. We now use two other (in addition to (4)) very important properties of $G(t, s)$ which were established in [9]; these are

$$(5) \quad G^{(k)}(a, s) > 0, \quad s \in [a, b],$$

$$(6) \quad (-1)^{n-k} G^{(n-k)}(b, s) > 0, \quad s \in (a, b].$$

Since

$$h^{(k)}(a) = \int_a^b G^{(k)}(a, s)Q(s)u(s) ds$$

it follows from (5) that

$$(7) \quad h^{(k)}(a) \in \mathcal{K}^\circ.$$

Similarly, using (6) it follows that

$$(8) \quad (-1)^{n-k} h^{(n-k)}(b) \in \mathcal{K}^\circ.$$

By Taylor's theorem about $t = a$ applied to the components y_i^j of y^j , we get that

$$y^j(t_j) = (y^j)^{(k)}(a) \frac{(t_j - a)^k}{k!} + R(\xi_{1j}, \dots, \xi_{mj}) \frac{(t_j - a)^{k+1}}{(k+1)!}$$

where

$$R(\xi_{1j}, \dots, \xi_{mj}) = \begin{pmatrix} (y_1^j)^{(k+1)}(\xi_{1j}) \\ \vdots \\ (y_m^j)^{(k+1)}(\xi_{mj}) \end{pmatrix}$$

where $a < \xi_{ij} < t_j$, $1 \leq i \leq m$, $j \geq 1$. Since

$$y^j(t_j) = \frac{(t_j - a)^k}{k!} \left[(y^j)^{(k)}(a) + R(\xi_{1j}, \dots, \xi_{mj}) \frac{(t_j - a)}{k+1} \right],$$

$(y^j)^{(k)}(a) \rightarrow h^{(k)}(a) \in \mathcal{K}^\circ$, and $R(\xi_{1j}, \dots, \xi_{mj}) \rightarrow h^{(k+1)}(a)$ as $j \rightarrow \infty$, it follows that $y^j(t_j) \in \mathcal{K}^\circ$ for j sufficiently large, which is a contradiction. Hence $t_0 \neq a$.

Similarly, if $t_0 = b$, then we use (8) and apply Taylor's theorem to the components of y^j to get a similar contradiction. Hence $N: \mathcal{P} \setminus \{0\} \rightarrow \mathcal{P}^\circ$. (Note that if we did not assume the zero conditions at b in the definition of $\mathcal{B}$, then our proof of $N: \mathcal{P} \setminus \{0\} \rightarrow \mathcal{P}^\circ$ breaks down here.)

Since $N: \mathcal{P} \setminus \{0\} \rightarrow \mathcal{P}^\circ$, $\mathcal{P}^\circ$ is nonempty. Therefore $\mathcal{P}$ is a solid cone and hence a reproducing cone in $\mathcal{B}$. Pick $u_0 \in \mathcal{P}^\circ$. Then $N: \mathcal{P} \setminus \{0\} \rightarrow \mathcal{P}^\circ$ implies N is u_0 -positive with respect to $\mathcal{P}$. The proof of this last statement appears in [5, p. 253]. For completeness we repeat this argument here. Since $Nu \in \mathcal{P}^\circ$, we have for sufficiently small $k_1 > 0$ that $Nu - k_1 u_0 \in \mathcal{P}$. On the other hand $u_0 \in \mathcal{P}^\circ$ so for $k_2 > 0$ sufficiently large, $u_0 - Nu/k_2 \in \mathcal{P}$. It follows that $k_1 u_0 \leq Nu \leq k_2 u_0$ with respect to $\mathcal{P}$, and so N is a u_0 -positive operator. The conclusions of this theorem then follow from Theorem 2 and the u_0 -positivity of N .

COROLLARY 2. *If for $1 \leq i, j \leq m$,*

$$q_{ij}(t)\delta_i\delta_j \geq 0, \quad t \in [a, b],$$

and q_{ij} is not identically zero on any subinterval of $[a, b]$, then the boundary value problem (2), (3) has a smallest positive eigenvalue Λ_0 , and Λ_0 is smaller than the modulus of any other eigenvalue of (2), (3). Further, there is an essentially unique eigenfunction $z_0(t)$ corresponding to Λ_0 and either z_0 or $-z_0$ is in $\mathcal{P}_1^\circ$. Hence, without loss of generality

$$\delta_i(z_0(t))_i > 0, \quad t \in (a, b), \quad 1 \leq i \leq m,$$

where $(z_0(t))_i$ is the i th component of $z_0(t)$.

PROOF. We will apply the above theorem with $\mathcal{K} = \mathcal{K}_1$ and $\mathcal{P} = \mathcal{P}_1$. It suffices to show that $Q(t)\mathcal{K}_1 \subseteq \mathcal{K}_1$ for $t \in [a, b]$, and for each $0 \neq u \in \mathcal{P}_1$, there exists $t_u \in [a, b]$ such that $Q(t_u)u(t_u) \in \mathcal{K}_1^\circ$.

Let $v \in K_1$. Then $\delta_i v_i \geq 0$, $1 \leq i \leq m$, and hence

$$\delta_i(Q(t)v)_i = \sum_{j=1}^m \delta_i \delta_j q_{ij}(t) \delta_j v_j \geq 0, \quad t \in [a, b], \quad 1 \leq i \leq m.$$

Hence $Q(t)K_1 \subseteq K_1$. If $0 \neq u \in P_1$, then there is a $t_1 \in [a, b]$ and a $j_0 \in \{1, \dots, m\}$ such that $\delta_{j_0} u_{j_0}(t_1) > 0$. Pick t_u sufficiently close to t_1 such that $\delta_{j_0} u_{j_0}(t_u) > 0$ and $\delta_i \delta_{j_0} q_{ij_0}(t_u) > 0$, $1 \leq i \leq m$. Then

$$\begin{aligned} \delta_i(Q(t_u)u(t_u))_i &= \sum_{j=1}^m \delta_i \delta_j q_{ij}(t_u) \delta_j u_j(t_u) \\ &\geq \delta_i \delta_{j_0} q_{ij_0}(t_u) \delta_{j_0} u_{j_0}(t_u) > 0 \end{aligned}$$

for $1 \leq i \leq m$. It follows that $Q(t_u)u(t_u) \in K_1^\circ$ and the proof is complete.

THEOREM 5. Assume $P(t)$ and $Q(t)$ satisfy the hypothesis of Theorem 4 and that $P(t) \leq Q(t)$ with respect to K , $t \in [a, b]$. Then the smallest positive eigenvalues λ_0 and Λ_0 of (1), (3) and (2), (3), respectively, satisfy $\Lambda_0 \leq \lambda_0$. Furthermore, if $\Lambda_0 = \lambda_0$ then

$$P(t)z_0(t) = Q(t)z_0(t), \quad t \in [a, b],$$

where $z_0(t)$ is as in Theorem 4.

PROOF. By the proof of Theorem 4, M and N are u_0 -positive with respect to $\mathcal{P}$. Further, by Theorem 4, the smallest positive eigenvalues λ_0 and Λ_0 exist.

It is easy to see that $P(t) \leq Q(t)$ with respect to K , $t \in [a, b]$, implies that $M \leq N$ with respect to $\mathcal{P}$. The conclusions of this theorem follow easily from Theorem 3.

For the cone K_1 , we can weaken the conditions on $P(t)$ by employing Theorem 1 and Corollary 1. This will enable us to prove more general results. Even in the case $Lx = x^{(n)}$ we believe that these results are new.

LEMMA 1. Assume

$$(9) \quad p_{ij}(t) \delta_i \delta_j \geq 0, \quad t \in [a, b],$$

$1 \leq i, j \leq m$, and there is a $t_0 \in [a, b]$ and an $i_0 \in \{1, \dots, m\}$ such that

$$(10) \quad p_{i_0 i_0}(t_0) > 0.$$

Then the eigenvalue problem (1), (3) has a least positive eigenvalue λ_0 which is a lower bound on the modulus of all eigenvalues of (1), (3). Furthermore, there is an eigenfunction $y_0(t)$ corresponding to λ_0 such that

$$\delta_i(y_0(t))_i \geq 0, \quad t \in [a, b], \quad 1 \leq i \leq m.$$

PROOF. Let $u \in P_1$ and consider

$$Mu(t) = \int_a^b G(t, s)P(s)u(s) ds.$$

It follows that

$$\delta_i(Mu)_i(t) = \int_a^b G(t, s) \sum_{j=1}^m \delta_i \delta_j p_{ij}(s) \delta_j u_j(s) ds.$$

Hence

$$\delta_i(Mu)_i(t) \geq 0, \quad t \in [a, b], \quad 1 \leq i \leq m,$$

and we have that $M: \mathcal{P}_1 \rightarrow \mathcal{P}_1$.

In order to apply Theorem 1, we now show that there is a $w \in \mathcal{P}_1$ and an $\varepsilon_0 > 0$ such that $Mw \geq \varepsilon_0 w$ with respect to the cone $\mathcal{P}_1$. To this end, define w by

$$w(t) = (t-a)^k(b-t)^{n-k}\delta_{i_0}e_{i_0}, \quad a \leq t \leq b,$$

where e_{i_0} is the unit vector in R^m in the i_0 direction. Note that

$$\delta_{i_0}(Mw)_{i_0}(t) = \int_a^b G(t, s)\delta_{i_0}\delta_{i_0}p_{i_0 i_0}(s)(s-a)^k(b-s)^{n-k}ds > 0$$

for $t \in (a, b)$ follows from (9) and (10). Using (5) and (6), it is easy to show that

$$\begin{aligned} \delta_{i_0}(Mw)_{i_0}^{(k)}(a) &> 0, \\ (-1)^{n-k}\delta_{i_0}(Mw)_{i_0}^{(n-k)}(b) &> 0. \end{aligned}$$

Hence $(Mw)_{i_0}(t)$ has a zero of exact order k at a and a zero of exact order $n-k$ at b . But $w_{i_0}(t)$ satisfies

$$\begin{aligned} \delta_{i_0}w_{i_0}(t) &> 0, \quad t \in (a, b), \\ w_{i_0}^{(i)}(a) &= 0, \quad 0 \leq i \leq k-1, \\ w_{i_0}^{(i)}(b) &= 0, \quad 0 \leq i \leq n-k-1, \\ \delta_{i_0}w_{i_0}^{(k)}(a) &> 0, \quad (-1)^{n-k}\delta_{i_0}w_{i_0}^{(n-k)}(b) > 0. \end{aligned}$$

It follows that for $\varepsilon_0 > 0$ sufficiently small

$$\delta_{i_0}(Mw)_{i_0}(t) \geq \varepsilon_0\delta_{i_0}w_{i_0}(t), \quad t \in [a, b].$$

Then it is easy to see that $Mw \geq \varepsilon_0 w$ with respect to the cone $\mathcal{P}_1$. By Theorem 1, Corollary 1, and Remark 1, the conclusions of this lemma follow.

THEOREM 6. *In addition to the assumption that the scalar equation $Lx = 0$ is $(j, n-j)$ -disconjugate on $[a, b]$, $k-1 \leq j \leq n-1$, assume*

- (a) *there is an $i_0 \in \{1, \dots, m\}$ and $t_0 \in [a, b]$ such that $p_{i_0 i_0}(t_0) > 0$,*
- (b) *$0 \leq p_{ij}(t)\delta_i\delta_j \leq q_{ij}(t)\delta_i\delta_j$, $t \in [a, b]$, $1 \leq i, j \leq m$, and $q_{ij} \not\equiv 0$ on any (nondegenerate) subinterval of $[a, b]$, $1 \leq i, j \leq m$.*

Then there exists a smallest positive eigenvalue λ_0 (which is a lower bound for the modulus of the eigenvalues) of (1), (3) and a corresponding eigenvector $y_0(t)$ where $y_0 \in \mathcal{P}_1$. The eigenvalue problem (2), (3) has a smallest positive eigenvalue Λ_0 (which is less than the modulus of any other eigenvalue of (2), (3)) and a corresponding essentially unique eigenfunction $z_0(t)$. We can pick $z_0 \in \mathcal{P}_1^\circ$, and then $z_0(t)$ satisfies

$$(11) \quad \delta_i(z_0)_i(t) > 0, \quad t \in (a, b), \quad 1 \leq i \leq m.$$

Furthermore, $\Lambda_0 \leq \lambda_0$, and $\Lambda_0 = \lambda_0$ iff $P(t) = Q(t)$ for $t \in [a, b]$.

PROOF. By Lemma 1, λ_0 and y_0 exist as in the statement of the theorem. By Corollary 2, the eigenvalue problem (2), (3) has a smallest positive eigenvalue Λ_0 and corresponding to Λ_0 there is an essentially unique eigenvector z_0 , and either

$z_0 \in \mathcal{P}_1$ or $-z_0 \in \mathcal{P}_1$. Pick $z_0 \in \mathcal{P}_1$. It can be shown that $M \leq N$. Hence by Theorem 3, we get that $\Lambda_0 \leq \lambda_0$.

Now assume $\Lambda_0 = \lambda_0$. Then by Theorem 3, $y_0(t)$ and $z_0(t)$ are linearly dependent on $[a, b]$. It follows from (1) and (2) that

$$P(t)z_0(t) = Q(t)z_0(t), \quad t \in [a, b].$$

Hence for $1 \leq i \leq m$,

$$\sum_{j=1}^m [q_{ij}(t) - p_{ij}(t)](z_0)_j(t) = 0, \quad t \in [a, b].$$

Then

$$\sum_{j=1}^m [(q_{ij}(t) - p_{ij}(t))\delta_i\delta_j](z_0)_j(t) = 0.$$

Since z_0 satisfies (11) and condition (b) holds, we get that

$$p_{ij}(t) = q_{ij}(t), \quad t \in (a, b), \quad 1 \leq i, j \leq m.$$

By continuity, it follows that $P(t) = Q(t)$ on the closed interval $[a, b]$.

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