ON $\mathcal{A}$ -SUBMODULES FOR REFLEXIVE OPERATOR ALGEBRAS

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ABSTRACT. In [2] the authors described all weakly closed $\mathcal{A}$ -submodules of $L(H)$ for a nest algebra $\mathcal{A}$ in terms of order homomorphisms of $\text{Lat } \mathcal{A}$. In this paper we prove that for any reflexive algebra $\mathcal{A}$ which is σ -weakly generated by rank-one operators in $\mathcal{A}$, every σ -weakly closed $\mathcal{A}$ -submodule can be characterized by an order homomorphism of $\text{Lat } \mathcal{A}$. In the case when $\mathcal{A}$ is a reflexive algebra with a completely distributive subspace lattice and $\mathcal{M}$ is a σ -weakly closed ideal of $\mathcal{A}$, we obtain necessary and sufficient conditions for the commutant of $\mathcal{A}$ modulo $\mathcal{M}$ to be equal to $\text{AlgLat } \mathcal{M}$.

Let H be a complex Hilbert space, and let $L(H)$ be the set of all bounded linear operators on H . The terminology and notation of this paper concerning nest algebras and reflexive subspaces of $L(H)$ may be found in [3].

Let $\mathcal{A}$ be a reflexive subalgebra of $L(H)$. Suppose that $E \mapsto \tilde{E}$ is an order homomorphism of $\text{Lat } \mathcal{A}$ into itself (i.e. $E \leq F$ implies $\tilde{E} \leq \tilde{F}$), where $\text{Lat } \mathcal{A}$ is the set of all invariant projections for $\mathcal{A}$. Then the set $\mathcal{M} = \{T \in L(H) : (I - \tilde{E})TE = 0 \text{ for all } E \in \text{Lat } \mathcal{A}\}$ is clearly a weakly closed $\mathcal{A}$ -submodule of $L(H)$.

J. A. Erdos and S. C. Power in [2] proved that any weakly closed $\mathcal{A}$ -submodule of $L(H)$ for a nest algebra $\mathcal{A}$ is of the above form. Here we prove that this is also true for any reflexive algebra $\mathcal{A}$ which is σ -weakly generated by rank-one operators in $\mathcal{A}$.

The following result is due to J. Kraus and D. R. Larson [3].

THEOREM 1. *Let $\mathcal{A}$ be a unital σ -weakly closed algebra which is σ -weakly generated by rank-one operators in $\mathcal{A}$. Then every σ -weakly closed left or right module of $\mathcal{A}$ is reflexive.*

THEOREM 2. *Let $\mathcal{A}$ be as in the above theorem, and let $\mathcal{M}$ be a σ -weakly closed $\mathcal{A}$ -submodule of $L(H)$. Then $\mathcal{M}$ has the form*

$$\mathcal{M} = \{T \in L(H) : (I - \tilde{E})TE = 0 \text{ for all } E \in \text{Lat } \mathcal{A}\},$$

where $E \mapsto \tilde{E}$ is some order homomorphism of $\text{Lat } \mathcal{A}$ into itself.

PROOF. For any $E \in \text{Lat } \mathcal{A}$, let $\tilde{E}$ be the orthogonal projection onto $[\mathcal{M}EH] = \bigvee \{\text{ran}(XE) : X \in \mathcal{M}\}$. Since $\mathcal{M}$ is an $\mathcal{A}$ -submodule, $\tilde{E}$ is invariant under $\mathcal{A}$ and clearly $E \mapsto \tilde{E}$ is an order homomorphism. Let $\mathcal{N} = \{T \in L(H) : (I - \tilde{E})TE = 0 \text{ for all } E \in \text{Lat } \mathcal{A}\}$. It is obvious that $\mathcal{N} \supseteq \mathcal{M}$. Conversely, if $T \in \mathcal{N}$, then $(I - \tilde{E})TE = 0$, so $[TEH] \subseteq [\tilde{E}H] = [\mathcal{M}EH]$ for any $E \in \text{Lat } \mathcal{A}$. Now for any

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$x \in H$, denote by E the orthogonal projection onto $[\mathcal{A}x]$. Then $E \in \text{Lat } \mathcal{A}$ and so $Tx \in T[EH] \subseteq \tilde{E}H = [\mathcal{M}EH] = [\mathcal{M}[\mathcal{A}x]] = [\mathcal{M}x]$. By the definition of reflexive subspace of $L(H)$, we know that $T \in \mathcal{M}$ since $\mathcal{M}$ is reflexive by Theorem 1. It follows that $\mathcal{M} \supseteq \mathcal{N}$ and so $\mathcal{M} = \mathcal{N}$. The proof is complete.

It is known from [4] that a commutative subspace lattice $\mathcal{L}$ is completely distributive iff the rank-one subalgebra of $\text{Alg } \mathcal{L}$ is σ -weakly dense in $\text{Alg } \mathcal{L}$. Thus we have the following result:

COROLLARY 3. *If $\mathcal{L}$ is a commutative and completely distributive subspace lattice, then every σ -weakly closed $\text{Alg } \mathcal{L}$ -submodule is of the form*

$$\{T \in L(H) : (I - \tilde{E})TE = 0 \text{ for all } E \in \mathcal{L}\},$$

where $E \mapsto \tilde{E}$ is an order homomorphism of $\mathcal{L}$ into itself.

COROLLARY 4. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 2. Then $\mathcal{M}$ is an ideal of $\mathcal{A}$ if and only if $\tilde{E} \leq E$ for every $E \in \text{Lat } \mathcal{A}$.*

LEMMA 5. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 2. Then $\mathcal{M}$ is an algebra if and only if $\tilde{\tilde{E}} \leq \tilde{E}$ for every $E \in \text{Lat } \mathcal{A}$.*

PROOF. Clearly if $\mathcal{M}$ is an algebra, then for any $E \in \text{Lat } \mathcal{A}$

$$\tilde{\tilde{E}}H = [\mathcal{M}[\mathcal{M}EH]] = [\mathcal{M}\mathcal{M}EH] \subseteq [\mathcal{M}EH] = \tilde{E}H$$

so $\tilde{\tilde{E}} \leq \tilde{E}$.

For the converse, if $S, T \in \mathcal{M}$, since

$$STE = S\tilde{E}TE = \tilde{\tilde{E}}S\tilde{E}TE = \tilde{\tilde{E}}STE$$

we have $(I - \tilde{E})STE = 0$, and so $ST \in \mathcal{M}$ which shows that $\mathcal{M}$ is an algebra.

LEMMA 6. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 2. Then $P \in \text{Lat } \mathcal{M}$ if and only if there exists some $E \in \text{Lat } \mathcal{A}$ such that*

$$\tilde{E} \leq P \leq E.$$

PROOF. If $\tilde{E} \leq P \leq E$ for some $E \in \text{Lat } \mathcal{A}$ and $T \in \mathcal{M}$, then $TP = TEP = \tilde{E}TEP = P\tilde{E}TEP = PTP$. Hence $P \in \text{Lat } \mathcal{M}$.

Conversely if $P \in \text{Lat } \mathcal{M}$, let E be the projection onto $[\mathcal{A}PH]$. Then $E \in \text{Lat } \mathcal{A}$, $E \geq P$ and

$$\tilde{E}H = [\mathcal{M}[\mathcal{A}PH]] = [\mathcal{M}\mathcal{A}PH] \subseteq [\mathcal{M}PH] \subseteq PH.$$

Thus $\tilde{E} \leq P \leq E$. The proof is complete.

The following theorems give a description of the commutant $C(\mathcal{A}, \mathcal{M})$ of $\mathcal{A}$ modulo $\mathcal{M}$ for an $\mathcal{A}$ -submodule $\mathcal{M}$. Recall that $C(\mathcal{A}, \mathcal{M})$ is defined by

$$C(\mathcal{A}, \mathcal{M}) = \{T \in L(H) : TA - AT \in \mathcal{M} \text{ for every } A \in \mathcal{A}\}.$$

THEOREM 7. *Let $\mathcal{A}$ be a reflexive algebra with commutative and completely distributive subspace lattice $\text{Lat } \mathcal{A}$, and let $\mathcal{M}$ be a σ -weakly closed $\mathcal{A}$ -submodule containing $\mathcal{A}$. Then $C(\mathcal{A}, \mathcal{M}) = \mathcal{M}$.*

PROOF. Obviously $C(\mathcal{A}, \mathcal{M}) \supseteq \mathcal{M}$. For any $T \in C(\mathcal{A}, \mathcal{M})$ and $E \in \text{Lat } \mathcal{A}$, since $\tilde{E} \in \mathcal{A}$, we have $(I - \tilde{E})(T\tilde{E} - \tilde{E}T)E = 0$, so $(I - \tilde{E})TE = 0$ since $\tilde{E} \geq E$, which implies $T \in \mathcal{M}$. The proof is complete.

COROLLARY 8. Let $\mathcal{A}$ be as in Theorem 7. If $H^1(\mathcal{A}, L(H)) = 0$, then $H^1(\mathcal{A}, \mathcal{M}) = 0$ for any σ -weakly closed $\mathcal{A}$ -submodule $\mathcal{M}$ containing $\mathcal{A}$, where $H^1(\mathcal{A}, \mathcal{M})$ is the first Hochschild cohomology space (see [1]) of $\mathcal{A}$ with coefficients in $\mathcal{M}$.

REMARK. In the paper [1] Christensen proved that if $\mathcal{A}$ is a reflexive algebra with commutative subspace lattice $\text{Lat } \mathcal{A}$, then for any σ -weakly closed algebra $\mathcal{M}$ containing $\mathcal{A}$, $H^1(\mathcal{A}, \mathcal{M}) = H^1(\mathcal{A}, L(H))$. In Corollary 8, $\mathcal{M}$ need not be an algebra but we need $\text{Lat } \mathcal{A}$ to be completely distributive.

LEMMA 9. Let $\mathcal{A}$ be as in Theorem 7, and let $\mathcal{M}$ be a σ -weakly closed ideal of $\mathcal{A}$ determined by an order homomorphism $E \mapsto \tilde{E}$. Then $PC(\mathcal{A}, \mathcal{M})|_{PH} \subseteq (P\mathcal{A}|_{PH})'$ for any $E \in \text{Lat } \mathcal{A}$, where $P = E - \tilde{E}$.

PROOF. See the proof of (ii) of Lemma 3.1 in [2].

DEFINITION 10. Let $\mathcal{A}$ be a reflexive algebra, and let $\mathcal{M}$ be a σ -weakly closed ideal of $\mathcal{A}$ determined by an order homomorphism $E \mapsto \tilde{E}$. $\mathcal{M}$ is said to have property (*) if $PC(\mathcal{A}, \mathcal{M})|_{PH} = \{\lambda I_{PH}\}$ for any $P = E - \tilde{E} > 0$.

Note that if $\mathcal{A}$ is a nest algebra and $\mathcal{M}$ is a σ -weakly closed ideal of $\mathcal{A}$, then $\mathcal{M}$ has property (*).

THEOREM 11. Let $\mathcal{A}$ and $\mathcal{M}$ be as in Lemma 9. Then $C(\mathcal{A}, \mathcal{M}) \supseteq \text{AlgLat } \mathcal{M}$. Furthermore, $C(\mathcal{A}, \mathcal{M}) = \text{AlgLat } \mathcal{M}$ if and only if $\mathcal{M}$ has property (*).

PROOF. Let $\mathcal{M}$ be determined by an order homomorphism $E \mapsto \tilde{E}$. By Corollary 4 we know that $\tilde{E} \leq E$ for any $E \in \text{Lat } \mathcal{A}$. First we prove $C(\mathcal{A}, \mathcal{M}) \supseteq \text{AlgLat } \mathcal{M}$. Let $T \in \text{AlgLat } \mathcal{M}$. For any $A \in \mathcal{A}$, we need to prove $TA - AT \in \mathcal{M}$. For all $E \in \text{Lat } \mathcal{A}$ and $G \leq E - \tilde{E}$, $\tilde{E} + G$ is invariant for T by Lemma 6. Thus $(E - \tilde{E})T(E - \tilde{E})$ leaves every subprojection of $E - \tilde{E}$ invariant. And so

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E})$$

for some scalar λ_E . Note that T is also in $\mathcal{A}$. Now we have

$$\begin{aligned} (I - \tilde{E})(TA - AT)E &= (I - \tilde{E})TEAE - (I - \tilde{E})AETE \\ &= (E - \tilde{E})T((E - \tilde{E}) + \tilde{E})A((E - \tilde{E}) + \tilde{E}) \\ &\quad - (E - \tilde{E})A((E - \tilde{E}) + \tilde{E})T((E - \tilde{E}) + \tilde{E}) \\ &= (E - \tilde{E})T(E - \tilde{E})A(E - \tilde{E}) - (E - \tilde{E})A(E - \tilde{E})T(E - \tilde{E}) \\ &= \lambda_E(E - \tilde{E})A(E - \tilde{E}) - (E - \tilde{E})A\lambda_E(E - \tilde{E}) \\ &= 0. \end{aligned}$$

Thus $TA - AT \in \mathcal{M}$ for all $A \in \mathcal{A}$, and hence $T \in C(\mathcal{A}, \mathcal{M})$. Now we prove the last statement of the theorem.

(i) Suppose that $C(\mathcal{A}, \mathcal{M}) = \text{AlgLat } \mathcal{M}$. From the proof of the first statement of this theorem, we know that for any $E \in \text{Lat } \mathcal{A}$, and any $T \in C(\mathcal{A}, \mathcal{M}) = \text{AlgLat } \mathcal{M}$,

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E}).$$

(ii) Suppose that $\mathcal{M}$ has property (*). For any $T \in C(\mathcal{A}, \mathcal{M})$ and $E \in \text{Lat } \mathcal{A}$, we need to prove

$$(I - (\tilde{E} + Q))T(\tilde{E} + Q) = 0$$

for any subprojection $Q \leq E - \tilde{E}$.

Since $E, \tilde{E} \in \mathcal{A}$, we have

$$\begin{aligned}(I - \tilde{E})(T\tilde{E} - \tilde{E}T)E &= 0, \\ (I - \tilde{E})(TE - ET)E &= 0.\end{aligned}$$

Thus

$$(I - \tilde{E})T\tilde{E} = 0$$

and

$$\begin{aligned}(I - \tilde{E})TE &= (E - \tilde{E})TE \\ &= (E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E}).\end{aligned}$$

Therefore

$$\begin{aligned}(I - (\tilde{E} + Q))T(\tilde{E} + Q) \\ &= (I - \tilde{E})TQ - QTQ \\ &= \lambda_E(E - \tilde{E})Q - \lambda_E Q(E - \tilde{E})Q = 0\end{aligned}$$

and hence $T \in \text{AlgLat } \mathcal{M}$. The proof is completed.

COROLLARY 12. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Lemma 9. If $(P\mathcal{A} \mid_{PH})' = \{\lambda I_{PH}\}$ for every $P = E - \tilde{E} > 0$, then $C(\mathcal{A}, \mathcal{M}) = \text{AlgLat } \mathcal{M}$.*

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