A UNIQUENESS THEOREM OF REFLECTABLE DEFORMATIONS OF A FUCHSIAN GROUP

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ABSTRACT. Let Γ be a Fuchsian group of signature $(p, n, m; \nu_1, \nu_2, \dots, \nu_n)$; $2p - 2 + m + \sum_{j=1}^n (1 - 1/\nu_j) > 0$. Let $I_1, I_2, \dots, I_m$ be a maximal set of inequivalent components of $\Omega \cap \hat{\mathbf{R}}$; Ω is the region of discontinuity and $\hat{\mathbf{R}}$ is the extended real line. Let ϕ be a quadratic differential for Γ . Let f be a solution of the Schwarzian differential equation $Sf = \phi$. If ϕ is reflectable, f maps each I_j into a circle C_j . For each $\gamma \in \Gamma$ there is a Moebius transformation $\chi(\gamma)$ such that $f \circ \gamma = \chi(\gamma) \circ f$. We prove that ϕ is determined by the homomorphism χ and the circles $C_1, C_2, \dots, C_m$.

Introduction. Let Γ be a Fuchsian group acting on the upper half plane U and having signature $\{p, n, m; \nu_1, \nu_2, \dots, \nu_n\}$; $n \geq 0$, $m \geq 0$ and $2p - 2 + m + \sum_{j=1}^n (1 - 1/\nu_j) > 0$. Let Ω be the region of discontinuity of Γ . Let $I_1, I_2, \dots, I_m$ be a maximal set of inequivalent components of $\Omega \cap \hat{\mathbf{R}}$; $\hat{\mathbf{R}}$ is the extended real line.

Let ϕ be a bounded reflectable quadratic differential for Γ (will be defined below). Let f be a meromorphic solution in U of the Schwarzian differential equation

$$Sf = \phi.$$

Then f is a local homeomorphism of U into the extended complex plane, and there exists a homomorphism χ of Γ into the Moebius group such that

$$f(\gamma(z)) = \chi(\gamma)(f(z)), \quad \gamma \in \Gamma, z \in U.$$

Moreover, f maps each interval I_j into a circle. We prove that ϕ is determined by the homomorphism χ and the circles on which $f(I_j)$ lie.

The theorem for the special case when Γ is a covering group of a compact Riemann surface was first proven by Poincaré. A generalization of the theorem by Poincaré for a group representing a Riemann surface with punctures was proven by Kra [4]. The theorem for the case when $m = 1$ and f is real on I_1 was proven by Gallo and Porter [1]. They have also proved injectivity of an "extended monodromy map" (see the remark) for the case when $m > 1$ in [2].

DEFINITIONS. The pair (f, χ) is a *deformation* of Γ if (i) f is a meromorphic local homeomorphism from U onto an open subset of the extended complex plane and (ii) χ is a homomorphism of Γ into the Moebius group such that

$$f \circ \gamma = \chi(\gamma) \circ f \quad \text{for all } \gamma \in \Gamma.$$

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If (f, χ) is a deformation of Γ ,

$$\phi = Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2$$

is a holomorphic *quadratic differential* for Γ ; that is,

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z), \quad z \in U, \gamma \in \Gamma.$$

It is a *bounded quadratic differential*, if

$$\sup\{(\operatorname{Im} z)^2|\phi(z)|; z \in U\} < \infty.$$

It is *reflectable*, if it has a holomorphic extension to $\Omega \cap \hat{\mathbf{R}}$ and it is real in $\Omega \cap \hat{\mathbf{R}}$.

A deformation (f, χ) is *bounded reflectable* if Sf is a bounded reflectable quadratic differential. It is easy to check that f maps each I_j into a circle if Sf is reflectable. We prove the following

THEOREM. *Let (f, χ) and (g, χ) be two bounded reflectable deformations of a Fuchsian group Γ of signature $\{p, n, m; \nu_1, \nu_2, \dots, \nu_n\}$; $2p - 2 + m + \sum_{j=1}^n (1 - 1/\nu_j) > 0$. If $f(I_j)$ and $g(I_j)$ lie on the same circle C_j for each j , $j = 1, 2, \dots, m$, then $f = g$.*

PROOF. We take

$$F(z) = \frac{(f(z) - g(z))^2}{f'(z)g'(z)}, \quad z \in U.$$

It is easy to see that F is holomorphic in U . Let $\chi(\gamma) = A_\gamma$ for $\gamma \in \Gamma$. We have, for $\gamma \in \Gamma$,

$$\begin{aligned} (1) \quad f \circ \gamma &= A_\gamma \circ f \\ g \circ \gamma &= A_\gamma \circ g. \end{aligned}$$

It is easy to check that for a Moebius transformation B ,

$$(2) \quad (B \circ f - B \circ g)^2 = (f - g)^2 B'(f)B'(g).$$

Using (1) and (2) one can easily prove that

$$F(\gamma(z))\gamma'(z)^{-2} = F(z), \quad z \in U, \gamma \in \Gamma;$$

that is, F is a (-2) differential for Γ .

The functions f and g map each $I_j \subset \Omega \cap \hat{\mathbf{R}}$ into a circle C_j . For each j , there is a Moebius transformation A_j mapping C_j onto $\hat{\mathbf{R}}$. Using (2) we can show that

$$\frac{(A_j \circ f - A_j \circ g)^2}{(A_j \circ f)'(A_j \circ g)'} = \frac{(f - g)^2}{f'g'} = F.$$

Since $A_j \circ f$ and $A_j \circ g$ are real on each I_j , F is real on each I_j . Since

$$F(\gamma(z))\gamma'(z)^{-2} = F(z), \quad \gamma \in \Gamma,$$

F is real on $\Omega \cap \hat{\mathbf{R}}$. Therefore, F extends to all of Ω .

Next we study the behaviour of F in a cusped region belonging to each puncture on Ω/Γ ; Ω/Γ is a surface of type $(P, 2n, 0)$, $P = 2p + m - 1$. We assume that $\gamma(z) = z + 1$ determines a puncture. $\chi(\gamma) = A_\gamma$ is also parabolic since Sf is a

bounded quadratic differential for Γ [3]. We replace f and g by $C \circ f$ and $C \circ g$, respectively, where C is a fixed Moebius transformation. Then A_γ is replaced by $C \circ A_\gamma \circ C^{-1}$ so that

$$C \circ A_\gamma \circ C^{-1}(z) = z + b, \quad b \neq 0.$$

Following Kra [3] we have, for $z \in U$,

$$f'(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

and

$$g'(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}; \quad a_0 = b_0 = b.$$

Hence

$$f(z) = bz + \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$$

and

$$g(z) = b(z) + \sum_{n=1}^{\infty} B_n e^{2\pi i n z}$$

for some A_n and B_n . Therefore, we have

$$(3) \quad F(z) = \frac{(\sum_{n=1}^{\infty} A_n e^{2\pi i n z} - \sum_{n=1}^{\infty} B_n e^{2\pi i n z})^2}{\sum_{n=0}^{\infty} a_n e^{2\pi i n z} \sum_{n=0}^{\infty} b_n e^{2\pi i n z}} = \sum_{n=2}^{\infty} d_n e^{2\pi i n z}$$

for some d_n .

Following Kra [5], we define

$$\text{red ord}_Q F = \frac{\text{ord}_Q F}{|\Gamma_Q|}, \quad Q \in \Omega,$$

where $|\Gamma_Q|$ is the order of the stabilizer Γ_Q of Q , and for each parabolic fixed point Q of Γ , we define

$$\text{red ord}_Q F = r$$

if the Fourier expansion of F at ∞ is

$$F(z) = \sum_{n=r}^{\infty} a_n e^{2\pi i n z}, \quad a_r \neq 0, \quad z \in \Omega.$$

Let $\overline{D}_0$ be a fundamental set for Γ ; that is, $\overline{D}_0$ is in one-to-one correspondence with the points in Ω/Γ and $\overline{D}_0 \subset \Omega \cup \{\text{parabolic fixed points}\}$. Since F is holomorphic in Ω , $\text{red ord } F \geq 0$ in Ω . By (3), $\text{red ord}_Q F \geq 2$ for each parabolic fixed point Q . Hence, in general,

$$\sum_{Q \in \overline{D}_0} \text{red ord}_Q F \geq 0;$$

notice that equality is possible when there is no parabolic fixed point.

F projects to a (-2) differential $\hat{F}$ on Ω/Γ . A direct calculation (as it is done in [5]) yields

$$\begin{aligned} & \sum_{Q \in \overline{D}_0} \text{red ord}_Q F \\ &= \sum_{X \in \Omega/\Gamma} \text{ord}_X \hat{F} - 2 \left(1 - \frac{1}{\nu(X)} \right); \nu(X) \text{ is the ramification order of } X \in \Omega/\Gamma \\ &= -2 \left[(2P - 2) + \sum_{j=1}^{2n} \left(1 - \frac{1}{\nu_j} \right) \right] \\ &= -4 \left[P - 1 + \sum_{j=1}^n \left(1 - \frac{1}{\nu_j} \right) \right] \\ &= -4 \left[2p - 2 + m + \sum_{j=1}^n \left(1 - \frac{1}{\nu_j} \right) \right] < 0, \end{aligned}$$

since $2p - 2 + m + \sum_{j=1}^n (1 - 1/\nu_j) > 0$ by our assumption. This proves that $F = 0$; that is, $f = g$.

REMARK. In [2] Gallo and Porter have defined an extended monodromy mapping

$$\Phi: \{\text{Bounded reflectable differentials}\} \rightarrow \text{Hom} \times K^m,$$

where K is the collection of circles in the extended complex plane $\hat{\mathbb{C}}$. For (f, χ) with $Sf = \phi$, a bounded reflectable differential for Γ , $\Phi(\phi) = (\chi, C_1, C_2, \dots, C_m)$ where C_j is the circle in which $f(I_j)$ lies.

They have proved that the quotient of the image space under inner automorphisms of $\text{Aut } \hat{\mathbb{C}}$ is a real analytic manifold and the corresponding mapping is injective. The injectivity of this mapping is the same as our result in the following sense.

Let (f, χ_1) and (g, χ_2) be two bounded reflectable deformations of Γ . Assume that $f(I_j) = C_j$ and $g(I_j) = C'_j$. If $\chi_2 = A\chi_1 A^{-1}$ and $C'_j = A \circ C_j$ for some $A \in \text{Aut } \hat{\mathbb{C}}$, then $g = A \circ f$; this follows from our results by taking

$$F = \frac{(g - A \circ f)^2}{g'(A \circ f)'}$$

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