THE EXISTENCE OF $K(\pi, 1)$ 4-MANIFOLDS WHICH ARE RATIONAL HOMOLOGY 4-SPHERES

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ABSTRACT. We construct rational homology 4-sphere manifolds with contractible universal cover.

The following question was asked by Thurston in Kirby's problem list [1, Problem 4.17].

PROBLEM. Can a homology or even a rational homology 4-sphere manifold ever be a $K(\pi, 1)$?

We will construct in this note a class of rational homology 4-spheres which are $K(\pi, 1)$'s. Unfortunately none of them are homology 4-spheres. The construction is based on the following well-known fact about $K(\pi, 1)$.

PROPOSITION. Suppose (V_i, U_i) , $i = 1, 2$, are connected space pairs such that V_i, U_i , are $K(\pi, 1)$'s and $i_*: \pi_1(U_i) \rightarrow \pi_1(V_i)$, $i = 1, 2$, the homomorphisms induced by the inclusion maps, are injective, and furthermore suppose there is a homeomorphism $h: U_1 \rightarrow U_2$. Then the gluing space

$$X = V_1 \cup_k V_2$$

is still a $K(\pi, 1)$.

PROOF. See any book on homotopy theory.

DEFINITION. A connected space pair (A, B) is called a good $K(\pi, 1)$ pair if A, B are $K(\pi, 1)$'s and $i_*: \pi_1(B) \rightarrow \pi_1(A)$ is injective.

In §1, we will construct two compact orientable 4-manifolds, W_1 and W_2 , with the same connected boundary $\partial W_1 = \partial W_2 = T_2 \times S^1$, where T_2 is the orientable surface of genus two, such that $(W_1, \partial W_1)$ and $(W_2, \partial W_2)$ are good $K(\pi, 1)$ pairs. And furthermore, W_1, W_2 subject to the following conditions (all homologies in this note are with rational coefficients except indicated otherwise):

- (i) $\chi(W_1) = 0$, $\chi(W_2) = 2$, and,
- (ii) $b_1(W_1) = 3$, $b_1(W_2) = 2$,
- (iii) $i_{1*}: H_1(\partial W_1) \rightarrow H_1(W_1), i_{2*}: H_1(\partial W_2) \rightarrow H_1(W_2)$ are surjective and $\ker(i_{1*}) \cap \ker(i_{2*} \phi_*) = 0$, for an appropriate homeomorphism $\phi: \partial W_1 \rightarrow \partial W_2$.

Gluing W_1, W_2 along the boundary via ϕ , the resulting 4-manifold $W = W_1 \cup_\phi W_2$ is then a rational homology 4-sphere. To see this, W is a $K(\pi, 1)$ because of the proposition, and

$$\chi(W) = \chi(W_1) + \chi(W_2) - \chi(\partial W_1) = 0 + 2 - 0 = 2.$$

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Hence, it suffices to show that $b_1(W) = 0$, since by Poincaré duality $\chi(W) = 1 - b_1(W) + b_2(W) - b_3(W) + 1 = 2 - 2b_1(W) + b_2(W)$. Using Mayer-Vietoris sequence for (W, W_1, W_2) , we obtain the exact sequence

$$H_2(W) \rightarrow H_1(\partial W_1) \xrightarrow{i_{1*} \oplus i_{2*} \phi_*} H_1(W_1) \oplus H_1(W_2) \rightarrow H_1(W) \rightarrow 0.$$

The conditions (ii) and (iii) above imply that $i_{1*} \oplus i_{2*} \phi_*$ is an isomorphism, and thus $b_1(W) = 0$.

NOTATION. T_g denotes the orientable closed surface of genus g . If $h: M \rightarrow M$ is a self-homeomorphism, $M \times_h I$ denotes the mapping torus of h , i.e., $M \times I / \sim: (x, 0) \sim (h(x), 1)$. If f is an involution on M , $M/\langle f \rangle$ denotes the orbit space.

1. The constructions of the manifolds W_1 and W_2 .

(1.1) *The manifold W_1 .* Let M be an orientable Haken 3-manifold with an incompressible boundary $\partial M = T_2$, and $H_1(M, \mathbb{Z})$ torsion free, such that the inclusion map $i_*: H_1(\partial M, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$ is surjective. Set $W_1 = M \times S^1$. Clearly by the construction $(W_1, \partial W_1)$ is a good $K(\pi, 1)$ pair, $\partial W_1 = T_2 \times S^1$, $\chi(W_1) = 0$, and $b_1(W_1) = 3$. Since $i_*: H_1(\partial M, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$ is surjective, $i_*: H_1(\partial W_1, \mathbb{Z}) \rightarrow H_1(W_1, \mathbb{Z})$ is still surjective. Let $V_1 = \ker(i_*: H_1(\partial W_1) \rightarrow H_1(W_1))$. Then it is well known that V_1 is a linear subspace of dimension 2 in $H_1(\partial M)$ included in $H_1(\partial W_1)$. Since $H_1(M, \mathbb{Z})$ is torsion free, V_1 is generated by the homology classes of two disjoint simple closed curves in ∂M included as a $\partial M \times \{\text{point}\}$ in $\partial M \times S^1 = \partial(W_1)$.

(1.2) *The manifold W_2 .* Manifold W_2 will be constructed such that it has a two-fold cover X , which is a T_3 -surface bundle over a punctured torus.

Let A be punctured torus consisting of two annuli B_1, B_2 symmetric about an axis l as in Figure 1, where β_1, β_2 are the cores of B_1, B_2 , $\beta_1 \cap \beta_2$ is in the axis l .

The manifold X is the T_3 -bundle over A with monodromy on the core β_i of B_i given by $h: T_3 \rightarrow T_3$, an orientation preserving homeomorphism of T_3 for $i = 1, 2$. Let $p: X \rightarrow A$ be the fibration map. From the construction we have

$$X|_{B_i} = (T_3 \times_h I) \times I \quad \text{for } i = 1, 2.$$

The monodromy of the T_3 -bundle over ∂A is given by $hhh^{-1}h^{-1} = \text{id}$, hence $X|_{\partial A} = T_3 \times S^1$. The long exact sequence of homotopy groups of fibration implies that X is a $K(\pi, 1)$, since both T_3 and A are $K(\pi, 1)$'s. Moreover, the commutativity of

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(T_3) & \longrightarrow & \pi_1(\partial X) & \longrightarrow & \pi_1(\partial A) \longrightarrow 1 \\ & & i_{1*} \downarrow & & i_* \downarrow & & i_{2*} \downarrow \\ 1 & \longrightarrow & \pi_1(T_3) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(A) \longrightarrow 1, \end{array}$$

together with injectivity of i_{1*} and i_{2*} show that $i_*: \pi_1(\partial X) \rightarrow \pi_1(X)$ is injective. Hence $(X, \partial X)$ is a good $K(\pi, 1)$ pair.

We now construct a fixed-point free orientation preserving involution f on X such that the quotient space $W_2 = X/\langle f \rangle$ satisfies the conditions quoted in the introduction.

Let g be the orientation preserving involution of A induced by π -rotation with respect to axis l . Each annuli B_i is preserved by g . Let τ be a fixed-point free orientation preserving involution of T_3 .

FIGURE 1

LEMMA 1. *If the monodromy h of the bundle X satisfies $h\tau h = \tau$, then there exists a fixed-point free orientation preserving involution f of X such that the following diagram commutes:*

$$\begin{array}{ccc}
 X & \xrightarrow{f} & X \\
 p \downarrow & & p \downarrow \\
 A & \xrightarrow{g} & A
 \end{array}$$

PROOF. Assume $h\tau h = \tau$. Then there is an involution on $T_3 \times_h I$ induced by $(\tau, \varepsilon): T_3 \times I \rightarrow T_3 \times I$, where $\varepsilon(t) = 1 - t$. Gluing two such involutions together yields an involution on $p^{-1}(\beta_1 \cup \beta_2)$ covering the involution g on $\beta_1 \cup \beta_2$. Thickening this up yields the required involution f on X such that it covers g involution on A . The fixed points of f must lie above fixed points of g , i.e., in one of the three fibers where the axis of g meets A . But restricted to any of these fibers f is conjugate to τ and hence is fixed point free. Q.E.D.

LEMMA 2. *If f is as constructed as above, and $W_2 = X/\langle f \rangle$, then $\partial W_2 = T_2 \times S^1$.*

PROOF. Consider the map q induced by p on the orbit space,

$$q: X/\langle f \rangle \rightarrow A/\langle g \rangle,$$

where $A/\langle g \rangle$ is a disk. Let y_0, y_1, y_2 be the fixed points of g , $y_0 \in B_1 \cap B_2$, $y_i \in B_i$, $i = 1, 2$; and let x_0, x_1, x_2 be the corresponding points in the disk $A/\langle g \rangle$. Lemma 1 shows that

$$q: X/\langle f \rangle - \bigcup_{i=0}^2 \{q^{-1}(x_i)\} \rightarrow A/\langle g \rangle - \{x_0, x_1, x_2\}$$

is a T_3 -bundle, say ξ . We claim that $\xi|_{\partial(A/\langle g \rangle)}$ is a T_3 -bundle over the circle $\partial(A/\langle g \rangle)$ with monodromy τ . To see this, let ν_0, ν_1, ν_2 be three based loops in $A/\langle g \rangle - \{x_0, x_1, x_2\}$ as in Figure 2(a); α_1, α_2 be the based loops in $A - \{y_0, y_1, y_2\}$ as in Figure 2(b). Choose a horizontal plane-field of the fiber bundle ξ , and choose the horizontal plane-field on the fiber bundle below (denoted by η)

$$P|: X - \bigcup_{i=0}^2 \{p^{-1}(y_i)\} \rightarrow A - \{y_0, y_1, y_2\}$$

to be the pull-back plane-field such that monodromies of the fiber bundle η over α_i 's are given by h with respect to the plane-field. The monodromies we are about to mention are with respect to these plane-fields. Clearly, the monodromy of ξ over γ_0 is τ . Since the loop $\gamma_0 \gamma_i$ lifts to a loop homotopic to α_i in $A - \{y_0, y_1, y_2\}$, the monodromy of ξ over $\gamma_0 \gamma_i$ is the same as the monodromy of η over α_i , which is h , for $i = 1, 2$. Hence the monodromy of ξ over $\gamma_1 \gamma_0 \gamma_2$ is $h^{-1} \tau h$, i.e.,

$$\partial W_2 \cong T_3 \times_\tau I.$$

There is the obvious projection $T_3 \times_\tau I \rightarrow T_3/\langle \tau \rangle = T_2$, which induces a fibration. The fibers are circles which are homologically of infinite order. Thus, from the classification of S^1 -fiber bundles, this implies that $T_3 \times_\tau I$ is homeomorphic to $T_2 \times S^1$. Q.E.D.

Clearly, $(W_2, \partial W_2)$ is a good $K(\pi, 1)$ pair. Also, we have

$$\chi(W_2) = \frac{1}{2} \chi(X) = \frac{1}{2} \chi(T_3) \chi(A) = \frac{1}{2} (-4) (-1) = 2.$$

Hence all it remains to show is the existence of homeomorphisms $h: T_3 \rightarrow T_3$, and $\phi: \partial W_1 \rightarrow \partial W_2$; such that the conditions (ii) and (iii) hold. We will establish this in the next section.

FIGURE 2

2. Homology calculations. The exact sequence of rational homologies coming from the fiber bundle $p: X \rightarrow A$ together with Lemma 1 yield the following commutative diagram:

$$\begin{array}{ccccccc}
 H_1(T_3) & \xrightarrow{1-h_*} & H_1(T_3) & \xrightarrow{i_*} & H_1(X) & \xrightarrow{p_*} & H_1(A) \cong \mathbb{Q} \oplus \mathbb{Q} \\
 & & \downarrow \tau_* & & \downarrow f_* & & \downarrow g_* \\
 H_1(T_3) & \xrightarrow{1-h_*} & H_1(T_3) & \xrightarrow{i_*} & H_1(X) & \xrightarrow{p_*} & H_1(A) \cong \mathbb{Q} \oplus \mathbb{Q}
 \end{array}$$

The homology $H_1(W_2)$ is identified via the transfer with the f_* -invariant subspace of $H_1(X)$. Since $g_*: H_1(A) \rightarrow H_1(A)$ is multiplication by (-1) , it follows that $H_1^{f_*}(X)$ is $\ker(\tau_* - \text{id})$ on $\text{coker}(h_* - \text{id})$. We claim that $i_*: H_1(\partial W_2) \rightarrow H_1(W_2)$ is onto. Actually, if $H_1(\partial W_2)$ is identified with $H_1(T_2) \oplus H_1(S^1)$ by Lemma 2, then $H_1(T_2) \cong \ker(\tau_* - \text{id})$ and the inclusion map i_* induces a projection from $\ker(\tau_* - \text{id})$ to $\ker(\tau_* - \text{id})$ on $\text{coker}(h_* - \text{id})$.

The following lemma shows that it suffices to find h such that $b_1(W_2) = 2$.

LEMMA 3. *If $b_1(W_2) = 2$, then there exists a homeomorphism $\psi: T_2 \rightarrow T_2$ such that the gluing map $\phi = \psi \times \text{id}: \partial W_1 \rightarrow \partial W_2$ has the property that*

$$i_{1*} \oplus i_{2*} \phi_*: H_1(\partial W_1) \rightarrow H_1(W_1) \oplus H_1(W_2)$$

is an isomorphism.

FIGURE 3

PROOF. It suffices to show that $i_{1*} \oplus i_{2*} \phi_*$ has no kernel, since $\dim H_1(\partial W_1) = 5$, $b_1(W_1) = 3$, and $b_1(W_2) = 2$. Recall that $V_1 = \ker(i_*: H_1(\partial W_1) \rightarrow H_1(W_1))$ is a 2-dimensional vector space contained in $H_1(T_2) \hookrightarrow H_1(\partial W_1)$. Moreover V_1 is generated by $[s_1], [s_2]$, where s_1, s_2 are two disjoint simple closed curves in T_2 . On the other hand, we have just shown that the inclusion $T_2 \times \{\text{point}\} \hookrightarrow \partial W_2 \hookrightarrow W_2$ induces an epimorphism $i_*: H_1(T_2) \rightarrow H_1(W_2)$. Thus $V_2 = \ker(i_*: H_1(T_2) \rightarrow H_1(W_2))$ is a two-dimensional vector space generated by, say $[t_1], [t_2]$, where t_1 and t_2 are simple closed curves in T_2 . The existence of the two simple closed curves in T_2 follows from the theorem that a homology class in $H_1(T_g, \mathbb{Z})$ is represented by a simple closed curve if and only if the class is primitive (see [2]). It is now easy to construct a homeomorphism $\psi: T_2 \rightarrow T_2$, such that $[s_i], \psi_*([t_i]), i = 1, 2$, generate $H_1(T_2)$. Consequently, $V_1 \cap \psi_* V_2 = \{0\}$ in $H_1(T_2)$, i.e., $\phi = \psi \times \text{id}: \partial W_1 \rightarrow \partial W_2$ is the required homeomorphism. Q.E.D.

We can now finish the construction by showing the existence of h and $\tau: T_3 \rightarrow T_3$ such that $h\tau h = \tau$, and $H_1(W_2) = \ker(\tau_* - \text{id})$ on $\text{coker}(h_* - \text{id})$ is of dimension two.

LEMMA 4. Put T_3 as in Figure 3(a), let τ be the π -rotation about the z -axis; and put T_3 as in Figure 3(b), let j be the $2\frac{\pi}{3}$ -rotation about z -axis. Let $h = \tau j \tau j^{-1}$. Then,

- (i) τ is an orientation-preserving fixed-point free involution of T_3 ,
- (ii) $h\tau h = \tau$,
- (iii) $\dim(\ker(\tau_* - \text{id}) \text{ on } \text{coker}(h_* - \text{id})) = 2$, i.e., $b_1(W_2) = 2$.

PROOF. Clearly (i), (ii) hold. To check (iii), choose a base of $H_1(T_3)$, $[a_i], [b_i]$, $i = 1, 2, 3$, as in Figure 3(a) and 3(b). τ_* induces the $(13)(46)$ permutation, and j_* induces the $(123)(456)$ permutation of the base $([a_1], [a_2], [a_3], [b_1], [b_2], [b_3])$. $(13)(46)(123)(456)((13)(46))^{-1} = (123)(456)(123)(456)$, i.e., $\tau_* j_* \tau_* j_*^{-1} = j_*$. Hence $h_* = j_*$. Thus $\text{coker}(h_* - \text{id})$ is a rank 2 spanned by, say a_2 and b_2 . This shows that $\tau_*: \text{coker}(h_* - \text{id}) \rightarrow \text{coker}(h_* - \text{id})$ is the identity. Consequently, $b_1(W_1) = 2$. Q.E.D.

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