WEAK CONDITIONS FOR GENERATION OF COSINE FAMILIES IN LINEAR TOPOLOGICAL SPACES

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ABSTRACT. Let A be a closed linear operator in a Banach space X . Weak conditions are found under which (I) the abstract Cauchy problem in X :

$$u''(t) = Au(t), \quad t \in \mathbb{R}; \quad u(0) = u_0, \quad u'(0) = u_1$$

has a unique solution for each u_0 and u_1 given in a dense subset Y of X , and (II) the set Y becomes a linear topological space where $A|_Y$ generates a continuous cosine family.

The conditions are satisfied for example by the generator of a strongly continuous or holomorphic semigroup in X .

1. INTRODUCTION

Let A be a closed linear operator in a Banach space X with norm $\|\cdot\|$, and consider the Cauchy problem for the second-order evolution equation in X :

$$(1) \quad u''(t) = Au(t), \quad t \in \mathbb{R};$$

$$(2) \quad u(0) = u_0 \quad \text{and} \quad u'(0) = u_1,$$

where $' = d/dt$. Concerning the operator-theoretical treatment, we have the condition due to Sova [9] and Da Prato and Giusti [3] (see Fattorini [5] and also [10]):

(A) The domain $D(A)$ of A is dense in X . There are positive constants M and ω such that the subset $\{z^2: \operatorname{Re} z > \omega\}$ of the complex plane C is included in the resolvent set $\rho(A)$ of A , and

$$(3) \quad \|(1/n!)(d/dz)^n z(z^2 - A)^{-1}x\| \leq M(\operatorname{Re} z - \omega)^{-n-1}\|x\|$$

holds for $x \in X$, $\operatorname{Re} z > \omega$ and $n = 0, 1, \dots$.

This means the existence of a strongly continuous cosine family $\{C(t): t \in \mathbb{R}\}$ in X : For $x \in X$ $t \rightarrow C(t)x$ is continuous;

$$(4) \quad C(t+t') + C(t-t') = 2C(t)C(t') \quad \text{for } t, t' \in \mathbb{R},$$

$$C(0) = I;$$

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and the generator A is defined by $Au = (d/dt)^2 C(t)u|_{t=0}$ with $D(A) = \{u \in X : t \rightarrow C(t)u \text{ is of class } C^2\}$. But (3) seems to be rather difficult to verify directly for a given differential operator (see Sova [9] and also [10]). This is very different from the case of the Hille-Yosida-Phillips theorem on semigroup generation. Yosida [12] solved (1)–(2) for a concrete wave equation by constructing a strongly continuous group in terms of the matrix operator $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$. Kiszyński [8] pointed out that (A) is equivalent to the condition:

(B) There exists a Banach space V including $D(A)$ and included continuously in X such that the operator $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ with domain $D(A) \times V$ generates a strongly continuous group in the Banach space $V \times X$.

Suggested by

$$\begin{pmatrix} 0 & I \\ A^{1/2} A^{1/2} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & A^{1/2} \\ A^{1/2} & 0 \end{pmatrix},$$

Fattorini [4, 5], Goldstein [6], and recently Yagi [11] solved (1)–(2) through the group generated by $A^{1/2} = i(-A)^{1/2}$, $i = (-1)^{1/2}$ (in the exact or a formal sense) for a class of operators. However it is still highly desirable to find out a convenient criterion for (3) or a condition weaker than (A) or (B).

The purpose of this paper is to deal with the problem (1)–(2) under the following condition on A :

(C) A is a closed linear operator with domain dense in X . There are constants $M > 0$, $0 < \theta < \pi$ and $k \geq -1$ such that the sector $|\arg z| \leq \theta$ in C is included in $\rho(A)$ and

$$(5) \quad \|(z - A)^{-1}x\| \leq M|z|^k \|x\|$$

holds for $x \in X$ and $|\arg z| \leq \theta$, where we assume, by replacement by A of $A - rI$ for $r > 0$ if necessary, that a neighborhood of 0 is included in $\rho(A)$.

The condition (C) is weaker than (A). In fact, under (A) a holomorphic semigroup in X is generated by A ; under (C) even a strongly continuous semigroup in X is not necessarily generated by A . In other words, any operator which generates a strongly continuous or holomorphic semigroup in X satisfies (C). As is well known, if A satisfies (C) with $k = -1$, fractional powers of $-A$ are defined by the Balakrishnan-Kato-Yosida approach. Guzman [7, Theorems 1.1 and 1.2] characterized the generator of a holomorphic semigroup in X having among other things a slow growth like $O(t^h)$, $-1 < h < 0$ as $t \downarrow 0$ in terms of a fractional power of $-A$ under (C), $-1 < k < 0$. It is therefore needless to look for examples of concrete differential operators illustrating (C).

The Cauchy problem (1)–(2) in X is to find a function $u(t)$ of $t \in R$ with values in X such that $u(t)$ belongs to $D(A)$ for $t \in R$, and $t \rightarrow u(t)$ is of class C^2 and satisfies (1) and (2). The main results of this paper are as follows.

Theorem 1. *If an operator A satisfies the condition (C), then there is a dense subset Y of X such that the problem (1)–(2) in X has a unique solution for each given u_0 and u_1 in Y .*

Theorem 2. Under (C), the set Y becomes a linear topological space and the restriction $A|_Y$ of A in Y generates a continuous cosine family in Y .

Moreover, Y is identified as an abstract Gevrey space associated with A of order a with $2 - 2\theta/\pi < a < 2$.

A problem of similar nature was discussed by Beals [1]: He dealt with the equation of first order $u'(t) = Au(t)$, $t \geq 0$ in X under a condition on A somewhat stronger than ours, and consequently solved "weakly hyperbolic" partial differential equations (see [2]). The idea of using nice linear topological spaces is also available to our problem (1)–(2) for the equation in X of "hyperbolic" type.

The operator valued function of $t \in R$ and $s > 0$:

$$C(t, s) = (2\pi i)^{-1} \int_Q e^{-s(-z)^b} \cos(t(-z)^{1/2})(z - A)^{-1} dz$$

defined with a suitable path Q and $b > \frac{1}{2}$ plays an essential role throughout. This integral was once used with $b = 1$ by Da Prato and Giusti [3, p. 359] to construct a strongly continuous cosine family in X under the condition (A). The main task in this paper is, roughly speaking, to show that for each $u_0 \in Y$, $C(t, s)C(0, s)^{-1}u_0$ is a unique solution of (1)–(2) with $u_1 = 0$ and next that for each $t \in R$, $C(t, s)C(0, s)^{-1}$ becomes a continuous operator $C(t): Y \rightarrow Y$ independently of $s > 0$ and $\{C(t): t \in R\}$ satisfies (4).

2. THE CAUCHY PROBLEM IN X

We shall begin this section with a discussion of the function $C(t, s)$ under the condition (C) and then give the proof of Theorem 1.

Let b satisfy $\frac{1}{2} < b < (2 - 2\theta/\pi)^{-1}$, and let $(-z)^b$ be the branch of the power function which is holomorphic off the nonnegative real axis and positive for $z < 0$. Let S be the sector $\{s \in C: |\arg s| < d\}$, where d is a constant with $0 < d < \pi/2 - b(\pi - \theta)$. Consider the function of $t \in R$ and $s \in S$

$$(6) \quad C(t, s) = (2\pi i)^{-1} \int_Q e^{-s(-z)^b} C(t; z)(z - A)^{-1} dz,$$

where $C(t; z) = \sum_{m=0}^{\infty} t^{2m} z^m / (2m)!$ and Q is an integral path which runs in $\rho(A)$ from $\infty e^{i(2\pi-\theta)}$ to $\infty e^{i\theta}$ avoiding the nonnegative real axis, for instance

$$\begin{aligned} & \{z: \arg z = 2\pi - \theta, |z| > p\} \cup \{z: |z| = p, \theta \leq \arg z \leq 2\pi - \theta\} \\ & \cup \{z: \arg z = \theta, |z| > p\} \end{aligned}$$

with a sufficiently small $p > 0$.

Since $\cos(\arg s + b \arg(-z))$ is not smaller than the constant $c = \cos(d + b(\pi - \theta))$ with $0 < c < 1$ for any $s \in S$ and $z \in Q$ with large $|z|$, we have the estimate

$$(7) \quad |e^{-s(-z)^b}| \leq e^{-c|s||z|^b}.$$

Thus using the estimate $|C(t; z)| \leq e^{|t||z|^{1/2}}$ together with (5), we obtain that the integral (6) converges uniformly on every bounded t -subinterval of R for each fixed $s \in S$. Hence $t \rightarrow C(t, s)$ is continuous from R to the bounded operators in X .

Next consider the associated function $S(t, s)$ defined by

$$S(t, s) = \int_0^t C(r, s) dr$$

for $t \in R$ and $s \in S$. Clearly we have

$$(8) \quad S(t, s) = (2\pi i)^{-1} \int_Q e^{-s(-z)^b} S(t; z)(z - A)^{-1} dz,$$

where $S(t; z) = \sum_{m=0}^{\infty} t^{2m+1} z^m / (2m+1)!$.

The following lemma is the most important step of this paper (cf. Da Prato and Giusti [3, Proposition 2]).

Lemma 1. For all $t, t' \in R$ and $s, s' \in S$,

$$(9) \quad C(t + t', s + s') + C(t - t', s + s') = 2C(t, s)C(t', s').$$

Proof. Let Q' be a path similar to Q , defined for $0 < p' < p$ and $(1 - 1/2b)\pi + d/b < \theta' < \theta$. Clearly an estimate similar to (7) also holds true for $s \in S$ and $w \in Q'$ with large $|w|$, and Q may be replaced by Q' in (6).

Using the resolvent equation $(z - A)^{-1} - (w - A)^{-1} = (w - z)(z - A)^{-1}(w - A)^{-1}$, we have

$$\begin{aligned} C(t, s)C(t', s') &= (2\pi i)^{-2} \int_Q \int_{Q'} e^{-s'(-w)^b} C(t'; w)(w - z)^{-1} dw \\ &\quad \cdot e^{-s(-z)^b} C(t; z)(z - A)^{-1} dz \\ &\quad + (2\pi i)^{-2} \int_{Q'} \int_Q e^{-s(-z)^b} C(t; z)(z - w)^{-1} dz \\ &\quad \cdot e^{-s'(-w)^b} C(t'; w)(w - A)^{-1} dw. \end{aligned}$$

The second term vanishes and the first equals

$$(2\pi i)^{-1} \int_Q e^{-(s+s')(-z)^b} C(t; z)C(t'; z)(z - A)^{-1} dz,$$

which gives (9) since

$$2C(t; z)C(t'; z) = C(t + t'; z) + C(t - t'; z). \quad \text{Q.E.D.}$$

Lemma 2. (i) For each $t \in R$, $C(t, s)$ and $S(t, s)$ are holomorphic functions of $s \in S$;

(ii) $C(t, s)X$ and $S(t, s)X$ are included in $D(A^n)$, $n = 1, 2, \dots$, and, in particular, for $t \in R$ and $s \in S$

$$(10) \quad \begin{cases} (\partial/\partial t)C(t, s) = AS(t, s), \\ (\partial/\partial t)^2 C(t, s) = AC(t, s). \end{cases}$$

Proof. The first assertion (i) is evident from (6) and (8) together with (7). Next differentiation in t of (6) is possible. In fact, the integral over Q of $e^{-s(-z)^b} S(t; z) z(z - A)^{-1}$ converges uniformly on every bounded t -subinterval of R for each $s \in S$. Since A is closed and $z(z - A)^{-1} = I + A(z - A)^{-1}$, the first equality of (10) is true. The remaining part of (ii) is clear. Q.E.D.

We shall deal with the function $U(s) = C(0, s)$ of $s \in S$. Taking $t = t' = 0$ in (9), we obtain

$$(11) \quad U(s + s') = U(s)U(s') \quad \text{for } s, s' \in S.$$

The following lemma was established by Beals [1, Lemmas 1 and 3] under a slightly stronger condition. But his method of proof is still available to our operator A satisfying satisfying (C).

Lemma 3. (i) For integer N with $k \leq N < k+1$ and $u \in D(A^{N+2})$, $U(s)u \rightarrow u$ in X as $s \rightarrow 0$ in S .

(ii) For each $s \in S$, $U(s)$ is one-one.

We can show here that our condition (C) on A guarantees uniqueness of the solution of (1)–(2) in X .

Proposition 4. Under the condition (C), the Cauchy problem (1)–(2) in X has at most one solution.

Proof. Let $u(t)$ be a solution of (1)–(2) with $u_0 = u_1 = 0$. For $t \geq 0$ and $s > 0$, consider the function

$$f(r) = C(t - r, s)u'(r) + AS(t - r, s)u(r), \quad 0 \leq r \leq t \quad (\text{cf. [4, p. 91]}).$$

Noting that A and $S(t, s)$ commute and using (10), we obtain $f'(r) = 0$, which together with $f(0) = 0$ implies $f(r) = 0$ for $0 \leq r \leq t$. In particular, $U(s)u'(t) = f(t) = 0$. Lemma 3(ii) implies $u'(t) = 0$, and hence $u(t) = 0$ for $t \geq 0$. Same reasoning for $t < 0$. Q.E.D.

Proof of Theorem 1. From (11) we obtain

$$(12) \quad U(s_1)X \supset U(s_2)X \quad \text{if } 0 < s_1 < s_2.$$

Set $Y = \bigcup_{s>0} U(s)X$. Since $D(A)$ is dense in X , $D(A^2) = A^{-1}D(A)$ is dense in $D(A)$, and inductively $D(A^{n+1})$ is dense in $D(A^n)$, hence in X . Thus $D(A^{N+2})$ is dense in X and, by Lemma 3(i), Y is dense in X .

Let u_0 and u_1 belong to Y . Then, by (12), u_0 and u_1 belong to $U(s)X$ for some $s > 0$. Setting

$$u(t) = C(t, s)U(s)^{-1}u_0 + S(t, s)U(s)^{-1}u_1$$

for $t \in R$ and using (10) and Proposition 4, we see that $u(t)$ is the desired solution of the problem (1)–(2) in X . Q.E.D.

3. THE COSINE FAMILY IN Y

In this section we shall show under the condition (C) that the function $C(t, s)U(s)^{-1}$ defines a cosine family in Y equipped with a suitable topology, and then give the proof of Theorem 2.

Since $U(s)^{-1}$ is closed for each $s > 0$, $U(s)X$ is a Banach space with the graph norm. The inclusion (12) implies that Y becomes a linear topological space when equipped with the inductive limit topology.

Lemma 5. For each $t \in R$ and $s_1, s_2 > 0$

$$(13) \quad U(s_1)C(t, s_2) = C(t, s_2)U(s_1) = C(t, s_1 + s_2).$$

If $0 < s_1 < s_2$, then $C(t, s_2)X \subset U(s_1)X$ and

$$(14) \quad U(s_1)^{-1}C(t, s_2) = C(t, s_2 - s_1).$$

Proof. Since $C(-t, s) = C(t, s)$, (9) with $t = 0$ or $t' = 0$ gives (13). The second assertion is a simple consequence of the first. Q.E.D.

Lemma 6. For $t \in R$ and $u \in U(s)X$ with $s > 0$ the following hold:

$$C(t, w)U(w)^{-1}u = C(t, s)U(s)^{-1}u \quad \text{for } 0 < w < s;$$

$$C(t, s)U(s)^{-1}u \text{ belongs to } U(w)X \quad \text{for } 0 < w < s.$$

Proof. From (11) and (13) we obtain that

$$\begin{aligned} C(t, s)U(s)^{-1}u &= C(t, w)U(s - w)U(s)^{-1}u \\ &= C(t, w)U(w)^{-1}U(w)U(s - w)U(s)^{-1}u \\ &= C(t, w)U(w)^{-1}u \end{aligned}$$

for $0 < w < s$. From (14) we see that

$$\begin{aligned} C(t, s)U(s)^{-1} &= U(w)U(w)^{-1}C(t, s)U(s)^{-1}u \\ &= U(w)C(t, s - w)U(s)^{-1}u \end{aligned}$$

for $0 < w < s$, which implies the second assertion. Q.E.D.

Lemma 6 shows that if $0 < w < s$, then $C(t, s)U(s)^{-1}$ maps $U(s)X$ into $U(w)X$, and $C(t, s)U(s)^{-1}$ and $C(t, w)U(w)^{-1}$ coincide on $U(s)X$. Thus for each $t \in R$, there is a unique continuous operator $C(t): Y \rightarrow Y$ such that

$$C(t)u = C(t, s)U(s)^{-1}u$$

for all $s > 0$ and $u \in U(s)X$.

Lemma 7. The family $\{C(t): t \in R\}$ satisfies (4).

Proof. For every $s > 0$ and $u \in U(s)X$,

$$C(0)u = C(0, s)U(s)^{-1}u = u$$

and therefore $C(0) = I$.

Next we have, by (9) and (14), that

$$\begin{aligned} \{C(t+t') + C(t-t')\}u &= C(t+t', s)U(s)^{-1}u + C(t-t', s)U(s)^{-1}u \\ &= 2C(t, s/2)C(t', s/2)U(s)^{-1}u \\ &= 2C(t, s/2)U(s/2)^{-1}C(t', s)U(s)^{-1}u \\ &= 2C(t)C(t')u. \quad \text{Q.E.D.} \end{aligned}$$

Recalling $S(t, s) = \int_0^t C(r, s) dr$, we find that Lemma 6 also holds true with $C(t, s)$ replaced by $S(t, s)$ and therefore that the associated sine family $\{S(t): t \in R\}$ can be defined similarly by

$$S(t)u = S(t, s)U(s)^{-1}u$$

for all $s > 0$ and $u \in U(s)X$.

Proof of Theorem 2. We see first from the equality

$$AU(s) = U(w)AU(s-w)$$

that A maps $U(s)X$ into $U(w)X$ for $0 < w < s$, and the restriction $A|_Y$ of A in Y is continuous.

Recalling (10), we obtain that $t \rightarrow C(t, s)$ is continuous and of class C^2 from R to the bounded operators in X . Thus $t \rightarrow C(t)$ is continuous and of class C^2 to the bounded operators mapping $U(s)X$ into $U(w)X$ for $0 < w < s$, and

$$\begin{cases} (d/dt)C(t) = A|_Y S(t), \\ (d/dt)^2 C(t) = A|_Y C(t) \end{cases}$$

hold for $t \in R$, in particular, $A|_Y = (d/dt)^2 C(t)|_{t=0}$.

Finally we shall compare Y with an abstract Gevrey space of order $a = 1/b$ with $1/2 < b < (2 - 2\theta/\pi)^{-1}$. Let $Y(h)$ be for each $h > 0$ the totality of $u \in X$ such that

$$\begin{cases} u \in D(A^n) & \text{for } n = 1, 2, \dots; \\ |u|_h = \sup_{n \geq 1} h^{an} \Gamma(an)^{-1} \|A^n u\| < \infty. \end{cases}$$

Clearly $Y(h)$ becomes a Banach space with norm $|\cdot|_h$ and satisfies

$$Y(h_1) \supset Y(h_2) \quad \text{if } 0 < h_1 < h_2,$$

and hence $\bigcup_{h>0} Y(h)$ becomes a Gevrey space of order a when equipped with the usual inductive limit topology.

According to Beals [1, Lemma 4] we can show that Y is identified as $\bigcup_{h>0} Y(h)$, and complete the proof of Theorem 2.

Lemma 8. *There is a positive number q such that*

$$U(s)X \subset Y(h) \quad \text{for } 0 < h < qs;$$

$$Y(h) \subset U(s)X \quad \text{for } 0 < s < qh,$$

where the inclusion mappings are continuous.

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