A GENERALIZATION OF THE SALEM TEST

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ABSTRACT. The Salem test is a criterion for uniform convergence of a Fourier series on the circle group. An improved test for convergence at a point and uniform convergence on an arbitrary set is obtained, as well as a necessary and sufficient condition for the convergence of the Fourier series of a function at a Lebesgue point.

The result with which we are concerned is a test for the convergence of the Fourier series of a summable function. It provides conditions for convergence at a point and for uniform convergence on a set.

This result generalizes the Salem test for uniform convergence [3, 7]. Bary's opinion of the Salem test was that it appears "at a first glance to be hardly suitable for application" [3, pp. 305–310], but it has served as the starting point of many other investigations. For example, various notions of generalized bounded variation, such as Salem's Φ -bounded variation [3] and Waterman's Λ -bounded variation [9, 10] arise directly from the ideas of the Salem test. Investigations into the everywhere convergence and uniform convergence of Fourier series under every change of variable [1, 2, 4, 5, 6, 8] also originated from these ideas.

We shall consider a summable real- or complex-valued function on the circle group T . For odd integers n , let

$$T_n(x, t) = \frac{f(x + t/n) - f(x + (t + \pi)/n)}{1} \\ + \frac{f(x + (t + 2\pi)/n) - f(x + (t + 3\pi)/n)}{3} + \dots \\ + \frac{f(x + (t + (n-1)\pi)/n) - f(x + (t + n\pi)/n)}{n}$$

and let $Q_n(x, t)$ be obtained from $T_n(x, t)$ by substituting $-t$ and $-\pi$ for t and π respectively.

The Salem test can then be stated as follows.

Salem Test. *If f is continuous on T and $T_n(x, 0)$ and $Q_n(x, 0)$ converge uniformly to 0, then the Fourier series of f converges uniformly.*

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Letting $S(f, x)$ denote the Fourier series of f , we can express our result as follows.

Generalized Salem Test. *If for an f in $L^1(T)$ we have*

$$(*) \quad \frac{1}{h} \int_0^h |f(x+t) + f(x-t) - 2f(x)| dt = o(1) \quad \text{as } h \rightarrow 0$$

and

$$(**) \quad \int_{\pi}^{2\pi} |T_n(x, t) + Q_n(x, t)| dt = o(1) \quad \text{as } n \rightarrow \infty,$$

then $S(f, x)$ converges to $f(x)$. If $(*)$ and $(**)$ hold uniformly on a set E and f is bounded on E , then $S(f, x)$ converges uniformly to $f(x)$ on E . Condition $(**)$ may be replaced by

$$(***) \quad \int_{\pi}^{2\pi} (T_n(x, t) + Q_n(x, t)) \sin t dt = o(1) \quad \text{as } n \rightarrow \infty,$$

and if $(*)$ holds at x , then $(***)$ is a necessary and sufficient condition that $S(f, x)$ converges to $f(x)$.

Remark 1. Conditions $(*)$ states that x is a (symmetric-) Lebesgue point of f . It would be of considerable interest to determine if condition $(*)$ can be replaced by the symmetric differentiability of $\int f$ at x , i.e., by

$$(*)' \quad \frac{1}{h} \int_0^h (f(x+t) + f(x-t) - 2f(x)) dt = o(1) \quad \text{as } h \rightarrow 0.$$

It is elementary that either $(*)$ or $(*)'$ uniformly on T is equivalent to continuity.

Remark 2. It is easily seen that the denominators $(2k-1)$, $k = 1, 2, \dots$, in the definitions of T_n and Q_n may be replaced by k .

Proof of the theorem. If $S_n(x)$ denotes the n th partial sum of the Fourier series of f at x and

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

then

$$S_n(f, x) - f(x) = o(1) + \frac{1}{\pi} \int_0^{\pi} \phi_x(t) \frac{\sin nt}{t} dt$$

uniformly in x . Assuming $(*)$ and integrating by parts,

$$\begin{aligned} \int_0^{\pi/n} \phi_x(t) \frac{\sin nt}{t} dt &= -n \int_0^{\pi/n} \left(\int_0^t \phi_x(u) du \right) \frac{\cos nt}{t} dt \\ &\quad + \int_0^{\pi/n} \left(\int_0^t \phi_x(u) du \right) \frac{\sin nt}{t^2} dt. \end{aligned}$$

The first term is

$$n \int_0^{\pi/n} o(1) dt = o(1)$$

and the second is

$$\int_0^{\pi/n} o(t) O\left(\frac{n}{t}\right) dt = o(1).$$

Hence

$$\int_0^{\pi/n} \phi_x(t) \frac{\sin nt}{t} dt = o(1)$$

at every x for which $(*)$ holds, and this is uniform on any set on which $(*)$ holds uniformly. We note that $(*)'$ would suffice here.

In estimating $S_n(f, x) - f(x)$, there is no loss in generality in assuming that n is odd. Letting $\sum'$ denote summation over even indices,

$$\begin{aligned} \int_{\pi/n}^{\pi} \phi_x(t) \frac{\sin nt}{t} dt &= \sum_{k=0}^{n-2} \int_{\pi/n}^{2\pi/n} \phi_x(t + k\pi/n) \frac{(-1)^k \sin nt}{t + k\pi/n} dt \\ &= \sum_{k=0}^{n-2'} \int_{\pi/n}^{2\pi/n} \left[\phi_x(t + k\pi/n) \frac{1}{t + k\pi/n} \right. \\ &\quad \left. - \phi_x(t + (k+1)\pi/n) \frac{1}{t + (k+1)\pi/n} \right] \sin nt dt \\ &= \int_{\pi/n}^{2\pi/n} \sum_{k=0}^{n-2'} [\phi_x(t + k\pi/n) - \phi_x(t + (k+1)\pi/n)] \frac{\sin nt}{t + k\pi/n} dt \\ &\quad + \int_{\pi/n}^{2\pi/n} \sum_{k=0}^{n-2'} \phi_x(t + (k+1)\pi/n) \left[\frac{1}{t + k\pi/n} - \frac{1}{t + (k+1)\pi/n} \right] \sin nt dt \\ &= \int_{\pi/n}^{2\pi/n} I dt + \int_{\pi/n}^{2\pi/n} J dt. \end{aligned}$$

Let $\Phi(t) = \Phi_x(t) = \int_0^t |\phi_x(u)| du$. Then

$$\begin{aligned} \left| \int_{\pi/n}^{2\pi/n} J dt \right| &= \left| \int_{\pi/n}^{2\pi/n} \sum_{k=0}^{n-2'} \phi_x(t + (k+1)\pi/n) \right. \\ &\quad \cdot \left. \left(\frac{1}{t + k\pi/n} - \frac{1}{t + (k+1)\pi/n} \right) \sin nt dt \right| \\ &\leq \frac{n}{\pi} \sum_{k=0}^{n-2'} \int_{\pi/n}^{2\pi/n} |\phi_x(t + (k+1)\pi/n)| \frac{1}{(k+1)(k+2)} dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{n}{\pi} \sum_{k=0}^{n-2} [\Phi((k+3)\pi/n) - \Phi((k+2)\pi/n)] \frac{1}{(k+1)^2} \\
&\leq \frac{n}{\pi} \sum_{k=0}^{n-3} [\Phi((k+3)\pi/n) - \Phi((k+2)\pi/n)] \frac{1}{(k+1)^2} \\
&\leq \frac{n}{\pi} \left[\sum_{k=1}^{n-3} \Phi((k+2)\pi/n) \left[\frac{1}{k^2} - \frac{1}{(k+1)^2} \right] + \Phi(\pi)/(n-2)^2 \right] \\
&= \frac{n}{\pi} \sum_{k=1}^N \cdots + \frac{n}{\pi} \left[\sum_{N+1}^{n-3} \cdots + \Phi(\pi)/(n-2)^2 \right].
\end{aligned}$$

The first term on the right is

$$n \left(\sum_{k=1}^N o\left(\frac{k}{n}\right) \frac{1}{k^3} \right) = o(1) \quad \text{as } n \rightarrow \infty,$$

assuming that $N/n \rightarrow 0$. The other term is

$$\leq cn(|f(x)| + \|f\|_1)(1/N^2 + 1/n^2).$$

Choosing $N \approx n^{2/3}$ we see that

$$\int_{\pi/n}^{2\pi/n} J \, dt = o(1) \quad \text{as } n \rightarrow \infty$$

for every x and that this holds uniformly on any set where $f(x)$ is bounded and $\Phi_x(h) = o(h)$ uniformly.

We observe now that

$$\begin{aligned}
&\int_{\pi/n}^{2\pi/n} I \, dt - \frac{1}{\pi} \int_{\pi}^{2\pi} (T_n + Q_n) \sin u \, du \\
&= \int_{\pi/n}^{2\pi/n} \sum_{k=0}^{n-2} [\phi_x(t + k\pi/n) - \phi_x(t + (k+1)\pi/n)] \\
&\quad \cdot \left[\frac{1}{t + k\pi/n} - \frac{1}{(k+1)\pi/n} \right] \sin nt \, dt \\
&\quad - \int_{\pi/n}^{2\pi/n} [f(x + t + (n-1)\pi/n) - f(x + t + \pi) \\
&\quad + f(x - t - (n-1)\pi/n) - f(x - t - \pi)] \sin nt \, dt
\end{aligned}$$

and this last term is dominated by $\omega_1(f, \pi/n)$, the integral modulus of continuity of f . Clearly the remaining term is $o(1)$ uniformly in x if f is continuous. Otherwise it is dominated by

$$\begin{aligned}
&n \sum_{k=0}^{n-2} [\Phi((k+2)\pi/n) - \Phi((k+1)\pi/n)] / (k+1)^2 \\
&\quad + n \sum_{k=0}^{n-2} [\Phi((k+3)\pi/n) - \Phi((k+2)\pi/n)] / (k+1)^2
\end{aligned}$$

and these sums are estimated just as was done for $\int_{\pi/n}^{2\pi/n} J dt$. Thus (***) is equivalent to

$$\int_{\pi/n}^{2\pi/n} I dt = o(1) \quad \text{as } n \rightarrow \infty$$

if (*) holds, implying that, under condition (*), condition (***) is necessary and sufficient for the convergence of $S(f, x)$ to sum $f(x)$. The estimate for the difference between $\int_{\pi/n}^{2\pi/n} I dt$ and $(1/\pi) \int_{\pi}^{2\pi} (T_n + Q_n) \sin u du$ is uniform on any set on which $f(x)$ is bounded.

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