MANIFOLDS AND STIEFEL-WHITNEY CLASSES

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(Communicated by Haynes R. Miller)

ABSTRACT. This paper constructs manifolds which generate the unoriented cobordism ring having a maximum number of vanishing high dimensional Stiefel-Whitney classes.

1. INTRODUCTION

This paper studies the ring $\mathcal{N}_*$ of unoriented cobordism classes of closed smooth manifolds. Thom [9] has shown that $\mathcal{N}_*$ is isomorphic to $\mathbb{Z}_2[x_i | i \neq 2^s - 1]$ the $\mathbb{Z}_2$ polynomial ring with generators x_i for all natural numbers i not of the form $2^s - 1$. The most common generators to work with are those represented by $\mathbb{R}P^{2n}$ in even dimensions and by certain Dold manifolds [2] in odd dimensions. The purpose of this paper is to provide a new set of generators whose nonzero Stiefel-Whitney numbers involve only low dimensional Stiefel-Whitney classes.

Stong and Yoshida [8] have shown that high dimensional nonbounding manifolds have high dimensional Wu classes. More precisely,

Proposition. *For any integer k there is an integer $N = N(k)$ so that a closed manifold M with Wu class $v = 1 + v_1 + \cdots + v_k$ bounds if $\dim M > N$.*

It is noted that no similar result holds for Stiefel-Whitney classes. What is proven in this paper is the following.

Theorem. *If M^n is a closed smooth n -manifold with Stiefel-Whitney class given by $w = 1 + w_1 + \cdots + w_k$, where k is less than or equal to the number of ones in the dyadic expansion of n , then M is decomposable in $\mathcal{N}_*$.*

This paper then, constructs n -manifolds M^n for each positive integer n not of the form $2^s - 1$ where each manifold M^n represents an indecomposable in $\mathcal{N}_*$ and has Stiefel-Whitney class $w = 1 + w_1 + \cdots + w_{k+1}$ where k is the number of ones in the dyadic expansion of n .

Received by the editors July 1, 1987 and, in revised form, December 2, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 57R20.

Key words and phrases. Indecomposable manifold, Stiefel-Whitney class.

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 0002-9939/89 \$1.00 + \$.25 per page

The author wishes to thank Robert E. Stong for his help in writing the dissertation from which this paper comes.

2. THE MAIN THEOREM

Let M^n be an n -manifold. Let $f: M \rightarrow BO$ be the classifying map for the tangent bundle over M , $f^*: H^*(BO) \rightarrow H^*(M)$ the associated homomorphism in cohomology with $\mathbb{Z}_2$ coefficients (as are all coefficients in this paper), and $\phi: H^*(BO(m)) \rightarrow H^*(B\mathbb{Z}_2^m)$ the monomorphism induced by the inclusion $i: \mathbb{Z}_2^m \hookrightarrow O(m)$. $H^*(B\mathbb{Z}_2^m)$ is isomorphic to $\mathbb{Z}_2[x_1, \dots, x_m]$ the $\mathbb{Z}_2$ polynomial ring with m one-dimensional generators, and the image of ϕ to the subring of symmetric polynomials. Recognizing that $H^i(BO) \rightarrow H^i(BO(m))$ is an isomorphism for $i \leq m$ one can make the following definitions. The total Stiefel-Whitney class w of M is given by

$$w = f^* \circ \phi^{-1} \left[\prod_{i=1}^m (1 + x_i) \right],$$

where m is sufficiently large, and the k th Stiefel-Whitney class w_k by

$$w_k = f^* \circ \phi^{-1} \left[\sum x_1 x_2 \cdots x_k \right]$$

where $\sum x_1 x_2 \cdots x_k$ is the smallest symmetric polynomial in $\mathbb{Z}_2[x_1, \dots, x_m]$ containing the term $x_1 x_2 \cdots x_k$. The total Wu class v and the k th s -class s_k are then defined by

$$v = \text{Sq}^{-1} w = f^* \circ \phi^{-1} \left[\prod_{i=1}^m (1 + x_i + x_i^2 + x_i^4 + \cdots) \right]$$

and

$$s_k = f^* \circ \phi^{-1} \left[\sum_{i=1}^m x_i^k \right].$$

Due to the fact that $s_k(BO(m))$ is a primitive element in $H^*(BO)$ for all k , the class of M^n is indecomposable in $\mathcal{N}_*$ if and only if $s_n(\mu_M) = 1$ (μ_M or μ will denote the fundamental homology class of M). Also, if $k = 2^r - 1$ for some r , then cupping with s_k is actually a cohomology operation into $H^n(M^n)$. This is seen as follows.

If $Q_k \in \mathcal{A}$ is defined inductively by $Q_0 = \text{Sq}^1$ and $Q_k = \text{Sq}^{2^k} Q_{k-1} + Q_{k-1} \text{Sq}^{2^k}$, then Q_k is a primitive element in $\mathcal{A}$ [3].

Proposition. $Q_k(x) = S_{2^k-1} \cup x$ for any $x \in H^{n-2^k+1}(M^n)$.

Proof. Define $\psi: \mathcal{A} \rightarrow H^*(MO)$ by $\psi(\alpha) = \alpha(u)$ where u is the Thom class in $H^*(BO)$. ψ is a monomorphism and so $\psi(Q_k) = p \cup u$, where p is a primitive in $H^*(BO)$ of dimension $2^k - 1$. Hence, $\psi(Q_k) = S_{2^k-1} \cup u$ since the s -classes are the only primitives in $H^*(BO)$. Then, by Poincaré duality, $Q_k(x) = s_{2^k-1} \cup x$ for any $x \in H^{n-2^k+1}(M^n)$.

Let $\alpha(n)$ denote the number of ones in the dyadic expansion of n , i.e., $\alpha(n) = r$ if $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_r}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. We are ready to prove the main theorem.

Proof. Let M^n be an n -manifold with $w = 1 + w_1 + \cdots + w_k$ where $k = \alpha(n)$. Write $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_k}$ where $0 \leq \alpha_1 < \cdots < \alpha_k$. We need to show that $s_n(\mu_M) = 0$.

Case 1. n even. Let $q = 1 + 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_{k-1}}$. Then

$$\begin{aligned} s_n &= f^* \circ \phi^{-1} \left[\sum_{i=1}^k (x_i^n) \right] = \text{Sq}^q \left[f^* \circ \phi^{-1} \left[\sum_{i=1}^k (x_i^{2^{\alpha_k-1}}) \right] \right] = v_q \cup s_{(2^{\alpha_k-1})} \\ &= s_{(2^{\alpha_k-1})} \left[f^* \circ \phi^{-1} \left(\sum x_1 x_2^{2^{\alpha_1}} \cdots x_r^{2^{\alpha_{k-1}}} \right) \right]. \end{aligned}$$

Since $s_{(2^{\alpha_k-1})}$ is a primitive operation,

$$s_{(2^{\alpha_k-1})}(v_q) = f^* \circ \phi^{-1} \left[\sum x_1^{2^{\alpha_k}} x_2^{2^{\alpha_1}} \cdots x_r^{2^{\alpha_{k-1}}} \right] = v_n = 0.$$

So $s_n(\mu_M) = 0$ and M is decomposable in $\mathcal{N}_*$.

Case 2. n odd. Let $q = n - 2^{\alpha_k} + 1 = 2^a + 2^{\alpha_{a+1}} + \cdots + 2^{\alpha_{k-1}}$ (so $\alpha_i + 1 = \alpha_{i+1}$ for $i < a$ and $\alpha_a + 1 < \alpha_{a+1}$). Then

$$s_n = \text{Sq}^q \left[f^* \circ \phi^{-1} \left[\sum_{i=1}^k (x_i^{2^{\alpha_k-1}}) \right] \right] = v_q \cup s_{(2^{\alpha_k-1})} = s_{(2^{\alpha_k-1})}(v_q).$$

Since $s_{(2^{\alpha_k-1})}$ is a primitive operation, $s_{(2^{\alpha_k-1})}(y^2) = 0$ for any $y^2 \in H^q(M)$. So we need only write v_q modulo terms which are squares. Hence,

$$\begin{aligned} s_{(2^{\alpha_k-1})}(v_q) &= s_{(2^{\alpha_k-1})} \left[f^* \circ \phi^{-1} \left(\sum x_1 x_2 x_3^2 x_4^4 \cdots x_{a+1}^{2^{a-1}} x_{a+2}^{2^{\alpha_{a+1}}} \cdots x_r^{2^{\alpha_{k-1}}} \right) \right] \\ &= f^* \circ \phi^{-1} \left[\sum x_1^{2^{\alpha_k}} x_2 x_3^2 \cdots x_r^{2^{\alpha_{k-1}}} \right] = v_n = 0. \end{aligned}$$

Hence, $s_n(\mu_M) = 0 \in \mathbf{Z}_2$. Thus, M is decomposable and we have shown that every indecomposable n -manifold M^n has a nonzero i th Stiefel-Whitney class w_i with $i > \alpha(n)$.

3. CONSTRUCTIONS

For each positive integer n not of the form $2^s - 1$, we construct an indecomposable n -manifold with $w_k = 0$ for all $k > \alpha(n) + 1$.

We will denote by $\mathbf{RP}(m_1, m_2, \dots, m_k)$ the manifold fibered over $\mathbf{RP}^{m_1} \times \mathbf{RP}^{m_2} \times \cdots \times \mathbf{RP}^{m_k}$ with fiber $\mathbf{RP}^{k-1}$:

$$\mathbf{RP}(m_1, \dots, m_k) = \mathbf{RP} \left(\begin{array}{c} \lambda_1 \oplus \cdots \oplus \lambda_k \\ \downarrow \\ \mathbf{RP}^{m_1} \times \cdots \times \mathbf{RP}^{m_k} \end{array} \right)$$

where λ_i is the pullback of the standard line bundle over $\mathbf{RP}^{m_i}$. Notice

$$\mathbf{RP}(m_1, \dots, m_k)$$

has dimension $n = [\sum_{i=1}^k (m_i + 1)] - 1$. These manifolds are nice to work with because both the Stiefel-Whitney class w and the s -number $s_n(\mu_{M^n})$ are easy to compute. If M is $\mathbf{RP}(m_1, \dots, m_k)$, c the first Stiefel-Whitney class of the standard line bundle over M , $\sum_{i_1 < \dots < i_j} a_{i_1} a_{i_2} \dots a_{i_j}$ the j th Stiefel-Whitney class of $\lambda_1 \oplus \dots \oplus \lambda_k$ where a_i is the pullback of the nonzero class in $H^1(\mathbf{RP}(m_i))$, then $w(M) = \prod_{i=1}^k (1 + a_i)^{m_i+1} (1 + a_i + c)$ with the relations $a_i^{m_i+1} = 0$ and $c^k = (\sum_{i=1}^k a_i) c^{k-1} + (\sum_{i < j} a_i a_j) c^{k-2} + \dots + a_1 a_2 \dots a_k$. And

$$s_n(\mu_M) = \sum_{i=1}^k \binom{n-1}{m_i} \pmod{2}.$$

Hence, M is indecomposable if and only if $\sum_{i=1}^k \binom{n-1}{m_i}$ is odd [7]. Now we are prepared to give the promised manifolds. First we will tackle the case of even dimensions.

Even case. If n is even, then n can be written uniquely as $n = 2^{\alpha_t} + 2^{\alpha_{t-1}} + \dots + 2^{\alpha_1} + 2^{\alpha_0}$, where $\alpha_t > \alpha_{t-1} > \dots > \alpha_1 > \alpha_0 > 0$.

$$M = \mathbf{RP}(2^{\alpha_t} - 1, 2^{\alpha_{t-1}} - 1, \dots, 2^{\alpha_1} - 1, 2^{\alpha_0-1} - 1, 2^{\alpha_0-1} - 1, 0)$$

is an indecomposable n -manifold with $w_k = 0$ for all $k > \alpha(n) + 1$. First we check the dimension of M . $\text{Dim}(M) = \sum_{i=1}^t (2^{\alpha_i} - 1 + 1) + 2(2^{\alpha_0-1} - 1 + 1) + 1 - 1 = n$. Second, to see that M is indecomposable, we show that

$$s = \sum_{i=1}^t \binom{n-1}{2^{\alpha_i} - 1} + 2 \binom{n-1}{2^{\alpha_0-1} - 1} + \binom{n-1}{0}$$

is odd. For $i \geq 1$ (i.e., $2^{\alpha_i} \geq 4$),

$$\binom{n-1}{2^{\alpha_i} - 1} = \binom{2^{\alpha_t} + \dots + 2^{\alpha_1} + 2^{\alpha_0-1} + 2^{\alpha_0-2} + \dots + 2 + 1}{2^{\alpha_i-1} + \dots + 2^{\alpha_0} + \dots + 1}$$

is even. Here we are using a lemma proven in Steenrod [6].

Lemma. Let $a = \sum_{i=0}^m a_i 2^i$ and $b = \sum_{i=0}^m b_i 2^i$ ($0 \leq a_i, b_i \leq 1$). Then

$$\binom{b}{a} = \prod_{i=0}^m \binom{b_i}{a_i} \pmod{2}.$$

Since the next term is also even and $\binom{n-1}{0}$ odd, s is odd and M indecomposable.

Third, we compute the Stiefel-Whitney class of M . $w = \prod_{i=1}^{t+3} (1 + a_i + c)$ where $a_{t+3} = 0$ and $c^{t+3} = (\sum a_i) c^{t+2} + \dots + (a_1 a_2 \dots a_{t+2}) c$. Hence, $w_{t+3} = c^{t+3} + (\sum a_i) c^{t+2} + \dots + (a_1 \dots a_{t+2}) c = 0$. So $w_k = 0$ if $k > t + 2 = \alpha(n) + 1$ and we have finished the case of even dimensions.

Odd case. In order to construct the desired manifolds with odd dimension, we need to use a fact proven by Stong [7].

Proposition. *If an indecomposable $(m + p)$ -manifold M' fibers over an m -manifold B' and $f: B \rightarrow B'$ is a degree-one map from an m -manifold B into B' , then the induced fibration $M^{m+p} = f^*(M')$ over B is an indecomposable manifold.*

We now move on to construct odd dimensional n -manifolds with $w = 1 + w_1 + \cdots + w_k$ where $k = \alpha(n) + 1$. Given n odd, write $n = 2^{\alpha_l} + 2^{\alpha_{l-1}} + \cdots + 2^{\alpha_1} + 2^{\alpha_0}$, where $\alpha_l > \cdots > \alpha_1 > \alpha_0 = 0$. Let $q = \min\{i | i \neq \alpha_i\}$, that is

$$\alpha_q = \min\{\alpha_i | \alpha_i > \alpha_{i-1} + 1\}.$$

As a first attempt, set

$$M' = \mathbf{RP}(2^{\alpha_l} - 1, \dots, 2^{\alpha_{q+1}} - 1, 2^{\alpha_q - 1}, 2^{\alpha_q - 1} - 1, 2^{\alpha_{q-1}} - 1, \dots, 2^{\alpha_0} - 1).$$

As before, we need to check the dimension of M' , that M' is indecomposable and that the top nonzero Stiefel-Whitney class of M' has low dimension. First, M' has dimension $\dim(M) = \sum_{i=0, i \neq q}^l (2^{\alpha_i}) + (2^{\alpha_q - 1} + 1) + (2^{\alpha_q - 1}) - 1 = n$. Second, we calculate

$$s = \sum_{\substack{i=0 \\ i \neq q}}^l \binom{n-1}{2^{\alpha_i} - 1} + \binom{n-1}{2^{\alpha_q - 1}} + \binom{n-1}{2^{\alpha_q - 1} - 1} + \binom{n-1}{2^{\alpha_0} - 1}.$$

Since $\binom{2^j}{2^{k+1}}$ is even for any nonnegative integers, we only need to consider the terms

$$\binom{n-1}{2^{\alpha_q - 1}} \quad \text{and} \quad \binom{n-1}{2^{\alpha_0} - 1}.$$

The first,

$$\binom{n-1}{2^{\alpha_q - 1}} = \binom{2^{\alpha_l} + \cdots + 2^{\alpha_q} + 2^{\alpha_q - 1} + \cdots + 2^{\alpha_1}}{2^{\alpha_q - 1}}$$

is even since $\alpha_q > \alpha_q - 1 > \alpha_{q-1}$. The second is odd since $2^{\alpha_0} - 1 = 0$ and $\binom{n-1}{0} = 1$. Thus s is odd and M' is indecomposable. Unfortunately, the Stiefel-Whitney class of M' is not quite what we want. Because w has a factor $(1 + a_q)^{2^{\alpha_q - 1} + 1} = (1 + a_q + a_q^{2^{\alpha_q - 1}})$ coming from $H^*(\mathbf{RP}^{2^{\alpha_q - 1}})$, it has high dimensional terms. It is true that every Stiefel-Whitney number divisible by w_k with $k > \alpha(n) + 1$ is zero, but by using the proposition proven at the start of this case, we can get w_k actually equal to zero for all $k > \alpha(n) + 1$. Let B' be the base space of M' and B the manifold fibered over

$$\begin{aligned} & \mathbf{RP}^{2^{\alpha_l} - 1} \times \cdots \times \mathbf{RP}^{2^{\alpha_{q+1}} - 1} \times \mathbf{RP}^{2^{\alpha_q - 1} - 1} \\ & \times \mathbf{RP}^{2^{\alpha_q - 1} - 1} \times \mathbf{RP}^{2^{\alpha_q - 1} - 1} \times \cdots \times \mathbf{RP}^0 \end{aligned}$$

with fiber $\mathbf{RP}(\lambda \oplus 1)$ where $\lambda \oplus 1$ is the pullback of the standard line bundle plus a trivial line bundle over the first $\mathbf{RP}^{2^{\alpha_q - 1} - 1}$ factor. Then, recognizing that the

bundle is isomorphic to the connected sum $\mathbf{RP}^n \# \mathbf{RP}^n$ [7], one lets $f: B \rightarrow B'$ be the map induced by

$$\begin{array}{ccc} \mathbf{RP}(\lambda \oplus 1) & & \\ f: \downarrow & \xrightarrow{\cong} & \mathbf{RP}^{2^{nq}-1} \# \mathbf{RP}^{2^{nq}-1} \rightarrow \mathbf{RP}^{2^{nq}-1}, \\ & & \mathbf{RP}^{2^{nq}-1-1} \end{array}$$

the map which collapses one summand to a point. Then f is a degree-one map and the pullback $M = f^*(M')$ is an indecomposable manifold fibered over

$$\begin{aligned} & \mathbf{RP}^{2^{\alpha_1}-1} \times \cdots \times \mathbf{RP}^{2^{\alpha_{q+1}}-1} \times \mathbf{RP}^{2^{nq}-1-1} \\ & \times \mathbf{RP}^{2^{nq}-1-1} \times \mathbf{RP}^{2^{nq}-1-1} \times \cdots \times \mathbf{RP}^0. \end{aligned}$$

Since the tangent bundle over $\mathbf{RP}^{2^{nq}-1}$ has $w = 1$, for all i , one has that $w_k = 0$ if

$$k \geq n - \left[\sum_{\substack{i=0 \\ i \neq q}}^t (2^{\alpha_i} - 1) + 2(2^{\alpha_q-1} - 1) \right] = n - (n - t - 3) = t + 3 = \alpha(n) + 2,$$

that is, $w = 1 + w_1 + \cdots + w_k$ with $k = \alpha(n) + 1$. We are then finished with constructions.

4. AFTERWORD

One would like to replace “is decomposable” with “bounds” in this theorem. However, one has then set about to solve a much harder problem. One would need to show all Stiefel-Whitney numbers vanish rather than just the s -number $s_n(\mu)$. It is however, conjectured that every n -manifold bounds if $w_k = 0$ for all $k > \alpha(n)$.

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