INFINITESIMAL PSEUDO-METRICS AND THE SCHWARZ LEMMA

M. KLIMEK

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ABSTRACT. In this paper we investigate the relationship between infinitesimal pseudo-metrics introduced by N. Sibony and K. Azukawa. Also we prove a version of the Schwarz lemma for plurisubharmonic functions.

INTRODUCTION

Let M be a complex manifold. Throughout the paper we shall be assuming that the dimension of M is n . Sibony in [8] and Azukawa in [1] introduced infinitesimal pseudo-metrics on the tangent bundle TM using families of bounded plurisubharmonic functions on M . We shall denote the pseudo-metrics by S_M and A_M respectively (the definitions of the pseudo-metrics are given in the next section). Both S_M and A_M contract holomorphic mappings and hence $C_M \leq S_M \leq K_M$ and $C_M \leq A_M \leq K_M$ where C_M and K_M denote the Carathéodory and Kobayashi infinitesimal pseudo-metrics respectively. The purpose of this paper is to study the relationship between S_M and A_M .

In order to be able to state the main results we need to recall the definition of an extremal plurisubharmonic function, originally introduced in [6] and then studied in [2, 4].

Let $\mathcal{P}(M, p)$ denote the family of all negative plurisubharmonic functions u on M such that for any holomorphic chart $\varphi: U \rightarrow \varphi(U) \subset \mathbb{C}^n$ where $p \in U \subset M$ and $\varphi(p) = 0$, the function $u \circ \varphi^{-1} - \log \|\cdot\|$ is bounded from above in a neighbourhood of 0. (By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{C}^n$.) Define

$$(1) \quad u_M(z, p) = \sup\{u(z) : u \in \mathcal{P}(M, p)\}.$$

It can be proved that $u_M(\cdot, p) \in \mathcal{P}(M, p)$. Moreover, if M is an open subset of $\mathbb{C}$ such that $u_M(z, p) \rightarrow 0$ as $z \rightarrow \partial M$, then $-u_M(\cdot, p)$ coincides with the Green function for M with pole at p . For further properties of the extremal function u_M and its applications in complex analysis see [6, 4, 2].

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Let $\mathcal{E}^2(\{p\})$ denote the family of all functions which are of class $\mathcal{E}^2$ in some neighbourhood of p .

We show the following.

Theorem 1. *If M is a complex manifold then $S_M \leq A_M$. If M is a Stein manifold then A_M is upper semicontinuous and hence $S_M^* \leq A_M$ (where the asterisk denotes the upper-semicontinuous regularization). If $\exp(2u_M(\cdot, p)) \in \mathcal{E}^2(\{p\})$ for some $p \in M$ then*

$$(2) \quad S_M(\xi) = A_M(\xi) = \langle \mathcal{L}(\exp(2u_M(\cdot, p)))(p)\xi, \xi \rangle^{1/2}, \quad \xi \in T_p M,$$

where $T_p M$ is the tangent space to M at p and $\langle \mathcal{L} \cdot, \cdot \rangle$ denotes the Levi form.

If P_M is an infinitesimal pseudometric on M and $p \in M$, the set $I_p(P_M) = \{v \in T_p M : P_M(v) < 1\}$ is called the indicatrix of P_M at p . It can be shown that the indicatrices of S_M are always convex (see [8]) and those of A_M are—in general—only starlike circular. Therefore, it is easy to furnish examples of manifolds M for which the two pseudo-metrics differ. For instance we can take a plurisubharmonic function $h: \mathbb{C}^n \rightarrow \mathbb{R}_+$ such that $h(\lambda z) = |\lambda|h(z)$ for $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$ and $M = \{z \in \mathbb{C}^n : h(z) < 1\}$ is not convex (e.g. $h(z_1, z_2) = \max\{|z_1|, |z_2|, 2\sqrt{|z_1 z_2|}\}$ for $(z_1, z_2) \in \mathbb{C}^2$). Then $A_M(0, \xi) = h(\xi)$ (see [1]) and hence $A_M \neq S_M$ on $T_0 M \equiv \mathbb{C}^n$. An example of a nonconvex M for which $S_M = A_M$ is provided by the open annulus $M = \{z \in \mathbb{C} : r < |z| < R\}$ where $r > 0$, $R > 0$. The equality follows from Theorem 1 because $\exp(2u_M(\cdot, p)) \in \mathcal{E}^2(M)$ (see [6]).

We also prove a version of the Schwarz lemma for plurisubharmonic functions. To state the result, we have to introduce some notation. Let $p \in M$ and let $\mathcal{S}(M, p)$ denote the family of all logarithmically plurisubharmonic functions $u: M \rightarrow [0, 1]$ such that $u(p) = 0$ and $u \in \mathcal{E}^2(\{p\})$. By $\mathcal{F}(M, p)$ we will denote the set of all plurisubharmonic functions u on M with the property that for each $z \in M \setminus \{p\}$ there exists a connected one-dimensional complex submanifold N of M such that $z, p \in N$ and the restriction of u to $N \setminus \{p\}$ is harmonic.

Theorem 2. *Let Ω be a relatively compact open connected subset of a Stein manifold M and let $u_\Omega = u_\Omega(\cdot, p)$. If $v \in \mathcal{S}(\Omega, p)$ then*

$$(3) \quad v \leq \exp(2u_\Omega) \quad \text{in } \Omega \setminus \{p\}.$$

Moreover if $\exp(2u_\Omega) \in \mathcal{E}^2(\{p\})$ then for all $\xi \in T_p M$

$$(4) \quad \langle \mathcal{L}v(p)\xi, \xi \rangle \leq \langle \mathcal{L}(\exp(2u_\Omega))(p)\xi, \xi \rangle.$$

If $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\xi \in T_p M$ then $v \equiv \exp(2u_\Omega)$. If $n = 1$ and the equality holds in (3) for one $z \in \Omega$ then $v \equiv \exp(2u_\Omega)$.

When Ω is equal to the unit disc, $\exp(2u_\Omega(z)) = |z|^2$ and hence the theorem reduces to the version of the Schwarz lemma proved by Sibony in [8].

It is easy to notice that the last conclusion of the theorem is not true for $n > 1$. For instance, if Ω is the open unit ball in $\mathbb{C}^2$, p is the origin and $v(z_1, z_2) = |z_1|^2 + a|z_2|^2$ (where $a \in [0, 1)$ is fixed) then $v \neq \exp(2u_\Omega) = \|\cdot\|^2$.

Note also that if $n = 1$, $\exp 2u_\Omega \in \mathcal{E}^2(\{p\})$. This is so, because $-u_\Omega$ is the generalized Green function for Ω with pole at p .

Assume now that $\Omega \subset M$ is such that $u = u_\Omega(\cdot, p)$ is a $\mathcal{E}^2$ -function on $\Omega \setminus \{p\}$ and $u(z) \rightarrow 0$ as z approaches the boundary of Ω . It is known that u satisfies the homogeneous Monge-Ampère equation $(dd^c u)^n = 0$ in $\Omega \setminus \{p\}$ (see [6]). If we also assume that $(dd^c u)^{n-1} \neq 0$ at each point of $\Omega \setminus \{p\}$, then—in view of [3]—there is a foliation of $\Omega \setminus \{p\}$ by one-dimensional complex manifolds with the property that the restriction of u to each of them is harmonic (an alternative proof can be found in [5]). As observed in [4], it follows from the maximum principle that if N is a leaf of the foliation, then $p \in \overline{N}$ and $\overline{N} \cap \Omega$ is a one-dimensional analytic subvariety of Ω . It would be interesting to know under what condition p is a regular point of $\overline{N} \cap \Omega$ for each leaf N of the foliation. (If this was the case, u would be a member of $\mathcal{F}(\Omega, p)$.) Under the assumption that Ω is a bounded strictly convex domain in $\mathbb{C}^n$, an affirmative answer follows from Lempert's study of the Kobayashi metric [7]. It has been conjectured by Demailly [4] that if Ω is strictly pseudoconvex then the above conditions are also met—i.e. u is smooth on $\Omega \setminus \{p\}$, $dd^c u$ has constant rank $n - 1$ in $\Omega \setminus \{p\}$ and the leaves of the associated foliation extend through p to complex submanifolds of Ω .

In the light of the above remarks, the assumption that $u_\Omega \in \mathcal{F}(\Omega, p)$ does not seem to be too restrictive.

1. THE FAMILIES OF FUNCTIONS $\mathcal{P}(M, p)$ AND $\mathcal{S}(M, p)$

In this section we shall establish the relationship between the two families of plurisubharmonic functions defined in the introduction.

The following lemma follows directly from the proof of a version of the Schwarz lemma obtained by Sibony in [8]. For the sake of completeness we give a proof here.

Lemma 1. *Let $D \subset \mathbb{C}$ be a neighbourhood of zero. If $u \in \mathcal{S}(D, 0)$ then*

$$(5) \quad u(z) = \frac{1}{4}(\Delta u)(0)|z|^2 + o(|z|^2), \quad \text{as } z \rightarrow 0.$$

Proof. In view of Taylor's formula for u at 0, $v(z) = (u(z)/|z|^2)^*$ is subharmonic in D and

$$\lim_{t \rightarrow 0^+} \frac{u(t\alpha, t\beta)}{t^2} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(0) \alpha^2 + \frac{\partial^2 u}{\partial x \partial y}(0) \alpha \beta + \frac{1}{2} \frac{\partial^2 u}{\partial y^2}(0) \beta^2$$

for any $\alpha + i\beta$ from the unit circle. Moreover, as closed line segments in $\mathbb{C}$ are not thin, the limit on the left hand side of the above equality is $v(0)$. By

taking $\alpha + i\beta$ equal to 1, i and $(1+i)/\sqrt{2}$ we conclude that

$$\frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial y^2}(0) \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y}(0) = 0.$$

Hence the Taylor expansion of u at 0 looks exactly as stated in our lemma.

Let M be a complex manifold and let $p \in M$. For $u \in \mathcal{E}^2(\{p\})$ one can define the Levi form $\langle \mathcal{L}u(p) \cdot, \cdot \rangle$ of u at p as follows. Let $\xi \in T_p M$ and let $\varphi: U \rightarrow \varphi(U) \subset \mathbb{C}^n$ be a chart on M in a neighbourhood U of p such that $\varphi(p) = 0$. If $(\zeta_1, \dots, \zeta_n) = d_p \varphi(\xi)$ then we put

$$\langle \mathcal{L}u(p) \xi, \xi \rangle = \sum_{i,j=1}^n \frac{\partial^2 (u \circ \varphi^{-1})(0)}{\partial z_i \partial \bar{z}_j} \zeta_i \bar{\zeta}_j.$$

It is easy to show that this definition is independent of the choice of φ .

Lemma 2. *If $u \in \mathcal{S}(M, p)$ then $\log \sqrt{u} \in \mathcal{P}(M, p)$. Moreover if $V \subset \mathbb{C}$ is a neighbourhood of the origin and $F: V \rightarrow M$ is a holomorphic mapping such that $F(0) = p$ and $F'(0) = \xi$ we have*

$$(6) \quad \langle \mathcal{L}u(p) \xi, \xi \rangle = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \in \mathbb{C}}} \frac{(u \circ F)(\lambda)}{|\lambda|^2}.$$

Proof. Without loss of generality we may assume that M is an open subset of $\mathbb{C}^n$ and $p = 0$. Since $\langle \mathcal{L}u(p) \xi, \xi \rangle = \frac{1}{4} \Delta(\lambda \rightarrow (u \circ F)(\lambda))|_{\lambda=0}$ (where $\lambda \in \mathbb{C}$), (5) implies that in a neighbourhood of the origin in $\mathbb{C}$

$$(7) \quad (u \circ F)(\lambda) = \langle \mathcal{L}u(0) \xi, \xi \rangle |\lambda|^2 + o(|\lambda|^2).$$

(6) is implied directly by (7). Also from (7), applied to $F(\lambda) = \lambda \xi$, we deduce that Taylor's expansion of u about the origin in $\mathbb{C}^n$ has the following form:

$$(8) \quad u(z) = \langle \mathcal{L}u(0) z, z \rangle + o(\|z\|^2).$$

The first conclusion of the lemma follows from (8).

It is interesting to notice that if $u \in PSH(\mathbb{C}^n) \cap \mathcal{E}^2(\{0\})$ is such that

$$(\dagger) \quad u(\lambda z) = |\lambda|^2 u(z)$$

for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$, then $u(z) = \langle \mathcal{L}u(0) z, z \rangle$ for all $z \in \mathbb{C}^n$ and hence $u^{1/2}$ is a seminorm. To see this, it is enough to apply the Laplace operator (with respect to λ) to both sides of $(\dagger)$. (See also the remarks following Theorem 1 in the Introduction.)

The families $\mathcal{S}(M, p)$ and $\mathcal{P}(M, p)$ have been used in the definitions of S_M and A_M respectively (see [8, 1, 2]). S_M is defined by the formula

$$(9) \quad S_M(\xi) = \sup\{\langle \mathcal{L}u(p) \xi, \xi \rangle^{1/2} : u \in \mathcal{S}(M, p)\}, \quad \xi \in T_p M.$$

For more information about the pseudo-metric S_M see [8, 9].

Assume $\xi \in T_p M$. Let $V \subset \mathbf{C}$ be a neighbourhood of the origin and let $F: V \rightarrow M$ be a holomorphic mapping such that $F(0) = p$ and $F'(0) = \xi$. Then we put

$$L_u[\xi] = \limsup_{\substack{\lambda \rightarrow 0 \\ \lambda \neq 0}} \frac{\exp(u \circ F)(\lambda)}{|\lambda|}$$

for any $u \in \mathcal{P}(M, p)$. It can be proved that the above definition is independent of the choice of F (see [1, 2]). (If $\exp(2u) \in \mathcal{C}^2(\{p\})$, it follows directly from (6).)

Following Azukawa [1, 2] we define

$$(10) \quad A_M(\xi) = \sup_{u \in \mathcal{P}(M, p)} \{L_u[\xi]\}, \quad \xi \in T_p M.$$

From the definition of the extremal function $u_M = u_M(\cdot, p)$ we conclude (as in [2]) that for $\xi \in T_p M$

$$(11) \quad A_M(\xi) = L_{u_M}[\xi].$$

2. SEMICONTINUITY OF A_M

In this section we shall prove that if M is a Stein manifold then A_M is upper semicontinuous.

Lemma 3. *If M is a Stein manifold then $u_M: M \times M \rightarrow [-\infty, 0)$ is upper semicontinuous.*

Proof. Since M is a Stein manifold, there exists a smooth plurisubharmonic function $\psi: M \rightarrow \mathbf{R}$ such that the set $M_c \stackrel{\text{def}}{=} \{z \in M: \psi(z) < c\}$ is relatively compact in M for each $c \in \mathbf{R}$. It follows from the definition of the extremal function u_M that the sequence $\{u_{M_j}\}_{j \in \mathbf{N}}$ is decreasing and $\lim_{j \rightarrow \infty} u_{M_j} = u_M$ (see also [2] and [4]). Furthermore, as each of the sets M_j is hyperconvex (in the sense of [4]), the functions $u_{M_j}: M_j \times M_j \rightarrow [-\infty, 0)$ are continuous by Theorem 4.14 in [4]. This means that the limit function u_M is upper semicontinuous.

Lemma 4. *If M is a complex manifold such that $u_M: M \times M \rightarrow [-\infty, 0)$ is upper semicontinuous then $A_M: TM \rightarrow [0, \infty)$ is also upper semicontinuous.*

Proof. Take $\xi_0 \in TM$. Let $\pi: TM \rightarrow M$ be the canonical projection (i.e. $\pi(\xi) = q \Leftrightarrow \xi \in T_q M$). Let $\varphi: U \rightarrow \varphi(U) \subset \mathbf{C}^n$ be a holomorphic chart in a neighbourhood U of $\pi(\xi_0) = p$. The chart φ generates a chart $\tilde{\varphi}$ on TM in the following way:

$$\begin{aligned} \tilde{\varphi}: \pi^{-1}(U) &\rightarrow \varphi(U) \times \mathbf{C}^n, \\ \tilde{\varphi}(\xi) &= ((\varphi \circ \pi)(\xi), a_1, \dots, a_n) \Leftrightarrow \xi = \sum_{j=1}^n a_j \frac{\partial}{\partial \varphi_j} \Big|_{\pi(\xi)}. \end{aligned}$$

Define $g: \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ by the formula $g(\lambda, z, a) = \lambda a + z$ and set $F(\lambda, \xi) = \varphi^{-1}(g(\lambda, \tilde{\varphi}(\xi)))$. Take a neighbourhood V_0 of ξ_0 and $r_0 > 0$ such that the mapping $\lambda \rightarrow F(\lambda, \xi)$ is well defined if $|\lambda| < r_0$ and ξ is fixed in V_0 . Let F_ξ denote this mapping. Clearly $F_\xi(0) = \pi(\xi)$. In particular $F_{\xi_0}(0) = p$. Notice that if

$$\xi = \sum_{j=1}^n a_j \frac{\partial}{\partial \varphi_j} \Big|_{\pi(\xi)},$$

then $(d_{\pi(\xi)}\varphi)(\xi) = (a_1, \dots, a_n)$. Therefore

$$d_0 F_\xi = (d_{\pi(\xi)}\varphi)^{-1} \circ d_0(\lambda \rightarrow g(\lambda, \tilde{\varphi}(\xi)))$$

and hence

$$F'_\xi(0) = d_0 F_\xi \left(\frac{d}{d\lambda} \Big|_0 \right) = (d_{\pi(\xi)}\varphi)^{-1}(a_1, \dots, a_n) = \xi.$$

Thus

$$(12) \quad A_M(\xi) = \limsup_{\substack{\lambda \rightarrow 0 \\ \lambda \in \mathbb{C}}} \frac{\exp u_M(F_\xi(\lambda), \pi(\xi))}{|\lambda|}, \quad \xi \in \pi^{-1}(U).$$

Suppose $A_M(\xi_0) < c$. By (12), there is a number $r \in (0, r_0)$ such that

$$\sup_{|\lambda|=r} (u_M(F_{\xi_0}(\lambda), \pi(\xi_0)) - \log |\lambda|) < \log c.$$

This means that

$$\sup_{|\lambda|=r} u_M(F_{\xi_0}(\lambda), \pi(\xi_0)) < \log rc.$$

Because of upper semicontinuity of u_M and continuity of F and π one can find a neighbourhood V of ξ_0 such that $V \subset V_0$ and

$$\sup_{|\lambda|=r} u_M(F_\xi(\lambda), \pi(\xi)) < \log rc, \quad \xi \in V.$$

Therefore

$$\sup_{|\lambda|=r} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c.$$

If ξ fixed in V , the function $(\lambda \rightarrow u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|)^*$ is subharmonic in the disc $\{\lambda: |\lambda| < r_0\}$. Thus by the maximum principle for subharmonic functions

$$\limsup_{\substack{\lambda \rightarrow 0 \\ \lambda \neq 0}} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c$$

for every $\xi \in V$. Consequently $A_M < c$ in V .

3. PROOF OF THE THEOREMS

The first statement of Theorem 1 follows directly from the definitions (9), (11) of the pseudo-metrics and from Lemma 2. Lemma 3 and Lemma 4 yield the second conclusion of the theorem.

The estimates (3), (4) in Theorem 2 follow from the definition of u_Ω and from Lemma 2.

Now assume that $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\xi \in T_p M$. Without loss of generality we may assume that M is submanifold of $\mathbb{C}^m$ for some m . Since Ω is relatively compact in M , it is contained in an open ball with centre at p and a positive radius r . It is clear that $\log \|z - p\| - \log r \leq u_\Omega(z)$ for all $z \in \Omega$. Let N be a connected one-dimensional complex submanifold of Ω such that $p \in N$ and $u|(N \setminus \{p\})$ is harmonic. Let $F: V \rightarrow M$ be a holomorphic parametrization of N in a neighbourhood of p , such that $0 \in V \subset \mathbb{C}$ and $F(0) = p$. By applying (6) and the lower estimate for u_Ω , we get

$$\langle \mathcal{L}(\exp 2u_\Omega)(p)F'(0), F'(0) \rangle \geq \lim_{\lambda \rightarrow 0} \frac{\|F(\lambda) - F(0)\|^2}{|r\lambda|^2} = \frac{\|F'(0)\|^2}{r^2} > 0.$$

Hence, (6) implies that

$$1 = \frac{\langle \mathcal{L}v(p)F'(0), F'(0) \rangle}{\langle \mathcal{L}(\exp 2u_\Omega)(p)F'(0), F'(0) \rangle} = \lim_{\lambda \rightarrow 0} \frac{(v \circ F)(\lambda)}{\exp(2u_\Omega \circ F)(\lambda)}.$$

Therefore the function $((v/\exp 2u_\Omega)|N)^*$ is subharmonic on N and attains its maximal value at p . Thus the maximum principle implies that $v = \exp(2u_\Omega)$ on N . As $u_\Omega \in \mathcal{F}(\Omega, p)$, the same equality holds in Ω .

If $n = 1$ and the equality holds in (3) for one point $z \in \Omega \setminus \{p\}$, then the subharmonic function $v/(\exp 2u_\Omega)$ has its maximum at z . Therefore—by the maximum principle—it is constant.

REFERENCES

1. K. Azukawa, *Two intrinsic pseudo-metrics with pseudoconvex indicatrices and star-like circular domains*, J. Math. Soc. Japan **38** (1986), 627–647.
2. —, *The invariant pseudo-metric related to negative plurisubharmonic functions*, Kodai Math. J. **10** (1987), 83–92.
3. E. Bedford and M. Kalka, *Foliations and complex Monge-Ampère equations*, Comm. Pure Appl. Math. **30** (1977), 543–571.
4. J. -P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. **194** (1987), 519–564.
5. M. Klimek, *Complex foliations generated by $(1, 1)$ -forms*, Proc. Amer. Math. Soc. **91** (1984), 601–606.
6. —, *Extremal plurisubharmonic functions and invariant pseudodistances*, Bull. Soc. Math. France **113** (1985), 231–240.
7. L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427–474.
8. N. Sibony, *A class of hyperbolic manifolds*, Ann. of Math. Stud., no. 100, Princeton Univ. Press, Princeton, N. J., 1981, pp. 357–372.
9. M. Suzuki, *The intrinsic metrics on the domains in $\mathbb{C}^n$* , Math. Rep. Toyama Univ. **6** (1983), 143–177.

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE DUBLIN, BELFIELD, DUBLIN 4, IRELAND